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# The metapopulation fitness criterion: proof and perspectives

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## Abstract

Metapopulation theory has recently been stirred by the development of a metapopulation persistence criterion ( $R_m$ ) quantifying the “lifetime dispersal success” of a newly colonized deme in a sparsely occupied metapopulation. No rigorous proof of this criterion in continuous time was available so far. Here, we show that this criterion can be mathematically justified from standard Jacobian arguments. A summary of the key elements of this proof emphasizes the assumptions under which this criterion is valid. Examples illustrate how to generally compute metapopulation fitness in continuous time models. The underlying assumptions behind the criterion are

discussed, as well as theoretical puzzles surrounding the concept of metapopulation fitness.

*Key words:* dispersal, fitness, invasibility, metapopulation, persistence criterion, viability

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## 1 Introduction

A topic of primary interest in population biology is the possibility of persistence (May, 1999). A biological system is said persistent when all its components are protected from deterministic extinction. This means that the boundaries at which one component gets extinct are repellent, or equivalently that each component of the system has positive growth rate when rare (for more details on definitions of persistence, see Jansen and Sigmund, 1998). This “protection” criterion is used extensively in population genetics, population dynamics and ecology (Roughgarden, 1979; Charlesworth, 1994; Caswell, 2001; Chesson, 2000).

For single-component systems, such as a single-species population, it is usually called a *viability* criterion. The viability of a population is commonly assessed by computing the basic reproduction ratio ( $R_0$ ) of the species when it is rare (Diekmann et al., 1990; Metz et al., 1992; Mylius and Diekmann, 1995; Caswell, 2001). In physiologically structured (e.g. age-structured) populations,

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this computation requires knowledge of the structure of the population (e.g.  
the age pyramid) and its fecundity function (the expected rate of offspring  
sired by one adult in each category).  $R_0$  is the fecundity averaged over all cat-  
egories, weighted according to the population structure (Charlesworth, 1994;  
Caswell, 2001).

In multiple-component systems, e.g. communities of several species, persis-  
tence can be studied for each species individually, with the background en-  
vironment (parameter values in the model) being set by the other species in  
the community.  $R_0$  can be calculated as before but assuming that the focal  
species has no impact on its environment because it is rare (Diekmann et al.,  
1998, 2001, 2003; Chesson, 2000). In this scenario, the persistence criterion  
is called an *invasibility* criterion, since it indicates whether the focal species  
(the invader) is able to invade the other species (the residents). This criterion  
constitutes a useful fitness measure to predict evolutionary trajectories under  
recurrent substitutions of mutants (Metz et al., 1992). Persistence of the whole  
community is achieved when each species can invade the others (a condition  
sometimes called “mutual invasibility”).

A difficulty arises when one wants to apply persistence criteria to spatially  
structured systems, such as metapopulations (Levins, 1969; Hanski and Gilpin,  
1997) and metacommunities (Holt, 1997; Leibold et al., 2004; Holyoak et al.,  
2005). Such systems are made of local subsystems (demes or patches) con-  
nected by dispersal. In this case, a raw use of  $R_0$  as a persistence criterion  
leads to tedious computations and often lacks simplicity (e.g. Lebreton, 1996).  
Dispersal multiplies the number of categories to keep a census of in order to  
compute the fecundity function of individuals. In addition, qualifying the rar-  
ity of a species is more difficult in a spatially structured system because a

species can be rare in different ways (for instance it can have very few indi-  
viduals in multiple demes or have a little more of them in a unique deme). As  
changes of deme state are not unidirectional (a local population can increase  
or decrease in abundance regardless of whether it increased or decreased in  
the recent past), a stable deme state distribution is a bit trickier to compute  
than a stable age distribution in a closed population (but see Diekmann et al.,  
1990 for complicated, though spatially unstructured, computations of  $R_0$ ).

The quest for a tractable persistence criterion and its application to metapop-  
ulations and metacommunities has been a recurrent subject of theoretical  
research (e.g. Chesson, 1984; Lebreton, 1996; Casagrandi and Gatto, 2002;  
Hastings and Botsford, 2006). One solution is to study the dynamics of state  
probabilities (e.g. Chesson, 1984; Metz and Gyllenberg, 2001; Casagrandi and  
Gatto, 2002) rather than the deterministic dynamics of each deme (as when  
one uses a logistic model of population growth to predict the trajectory of the  
mean population abundance in each deme, e.g. Hastings and Botsford, 2006).

This kind of formalism is often employed in statistical physics (van Kampen,  
2007). Focusing on state probabilities without monitoring the state of each  
deme is justified as long as the number of demes is large (effectively infinite).  
Under this assumption, mean values of stochastic variables equal their average  
values over all the demes studied.

In two recent papers, Metz and Gyllenberg introduced the metapopulation  
closest equivalent to the single-population  $R_0$  (Gyllenberg and Metz, 2001;  
Metz and Gyllenberg, 2001). Their fitness measure, called  $R_m$ , measures the  
*lifetime dispersal success* of occupied demes in a metapopulation:  $R_m$  is com-  
puted as the average number of dispersers produced by a typical colonized  
deme, from its colonization by a unique immigrant up to its extinction. The

proposed method solves the problem of rarity in a metapopulation by comput-  
68 ing the "typical rarity" pattern of a species based on its dispersal parameters.

This persistence measure was already mentioned in a seminal paper of Ches-  
70 son (1981), in the context of discrete-time metapopulation models, and proved  
in a following paper (Chesson, 1984). Its application to continuous-time mod-  
72 els was proposed almost simultaneously by Metz and Gyllenberg (Gyllenberg  
and Metz, 2001; Metz and Gyllenberg, 2001) and by Casagrandi and Gatto  
74 (2002), albeit in different contexts. Metz and Gyllenberg (2001) emphasized  
the use of  $R_m$  as an invasion criterion, whereas Casagrandi and Gatto (2002)  
76 used it as a viability criterion. Interestingly, Casagrandi and Gatto (2002)  
derived  $R_m$  from feasibility constraints (i.e. as a necessary condition for the  
78 existence of an interior invariant state of the metapopulation), whereas Metz  
and Gyllenberg (2001) directly defined it as a persistence property. Casagrandi  
80 and Gatto (2002) proved that both conditions (existence of a feasible invariant  
probability measure and viability of the metapopulation near total extinction)  
82 are equivalent for their particular metapopulation model, in which there are  
no deme extinctions. From Markov chain theory, these conditions are indeed  
84 linked: the viability criterion means that the Markov chain is irreducible on  
non-empty states (i.e. there is a path leading from the state where the fo-  
86 cal species is rare to a state where it is not), while the feasibility criterion  
indicates that the Markov chain admits an invariant measure. The viability  
88 criterion thus implies the feasibility criterion, but the reverse should not hold  
in general. Hence, it is probably more honest to attribute the paternity of the  
90  $R_m$  criterion to Chesson (1984) and Metz and Gyllenberg (2001), since the  
case studied by Casagrandi and Gatto (2002) is a particular case of the one  
92 envisaged by Metz and Gyllenberg (2001).

Whereas Chesson (1984) proved the validity of the  $R_m$  criterion for discrete-  
94 time Markovian metapopulation dynamics (under specific assumptions), no  
such proof has been provided for continuous-time metapopulation models.  
96 Gyllenberg and Metz (2001) seemed to imply that the  $R_m$  criterion is always  
true in continuous-time metapopulation models, and only mentioned that “in  
98 most cases of interest, positivity arguments” should suffice to ensure this (Gyl-  
lenberg and Metz, 2001). However, this has not been checked in the papers  
100 presenting the basic metapopulation model (Gyllenberg and Metz, 2001; Metz  
and Gyllenberg, 2001). And, accordingly, positivity of a matrix is a mathe-  
102 matical property with little biological meaning. This article has two primary  
goals: (i) remedying the absence of proof for the continuous-time version of the  
104 criterion and (ii) assessing whether the  $R_m$  criterion of Metz and Gyllenberg  
depends on specific assumptions.

106 We use a general but sensible model of metapopulation with continuous-time  
dynamics (similar to the one in Gyllenberg and Metz, 2001; Metz and Gyl-  
108 lenberg, 2001; Casagrandi and Gatto, 2002). Classically, persistence analysis  
of a population or metapopulation studied in continuous time (i.e. through  
110 differential equations) is carried out by linearization of the dynamical equa-  
tions. Persistence holds as soon as the Jacobian matrix qualifying the trivial  
112 (extinction) equilibrium admits at least one eigenvalue with positive real part.  
This method of investigation has solid mathematical background (e.g. Horn  
114 and Johnson, 1991).

Using these standard Jacobian methods, we find necessary and sufficient condi-  
116 tions for persistence. We discuss their biological meaning and their connection  
to Metz and Gyllenberg’s  $R_m$  criterion. Our main finding is that the latter  
118 verbal criterion is acceptable in all continuous-time Markov process metapop-

ulation models. We provide a recipe to obtain analytical expressions of the  
120 criterion for a broad class of models, and illustrate it with examples. Com-  
bined with Chesson (1984)’s proof for discrete-time Poisson processes, our  
122 results suggest that the  $R_m$  criterion can be applied to typical “memoryless”  
metapopulations (i.e. models with no time lag between population abundance  
124 and recruitment).

Given the similarity between the viability criterion in a metapopulation and  
126 the invasion criterion in a metacommunity, we present results for single-species  
systems (viability) only. The extension of our results to invasion are straight-  
128 forward, as discussed in the last section. We first summarize the issue and  
model chosen by Metz and Gyllenberg (2001), using a very simple structured  
130 metapopulation.

## 2 Rationale of the $R_m$ criterion

### 132 2.1 A simple toy model

Consider a closed spatially implicit metapopulation, formed by an infinity of  
134 demes, in which every deme has two potential microsites, and thus a popula-  
tion size of 0, 1 or 2 individuals. At birth, individuals may choose to disperse  
136 and, if they do so, they enter a disperser pool. Only individuals that have  
settled in microsites can reproduce (asexually) and there are no explicit life  
138 stages except the potential disperser stage between birth and settlement.

Let  $\mu_i$ ,  $\lambda_i$  and  $\gamma_i$  be the rates of per capita mortality, per capita birth, and  
140 deme extinction respectively, in a deme containing  $i$  individuals ( $i$  equals 1



or 2). Newborns in a deme with  $i$  settled individuals choose to disperse with  
142 probability  $d_i$ . Let  $\delta$  be the density of dispersers in the disperser pool, and  
assume that this density is homogeneous in space and can be described using  
144 a deterministic equation in continuous time. Let  $\alpha$  and  $\mu_D$  be the rates at which  
dispersers try to settle in a deme and the rate at which they die, respectively.  
146 Dispersers arriving in a deme with  $i$  settled individuals stay with probability  
 $s_i$ . A disperser that does not stay in a deme goes back to the disperser pool.

148 Intuitively,  $s_2$  must be zero so that fully occupied demes cannot receive new  
migrants, and  $d_2$  must be one, so that newborns in fully occupied demes always  
150 disperse. This model is summarized in Fig. 1.

Let  $p_k(t)$  be the proportion of demes that are occupied by  $k$  individuals at time  
152  $t$ . The four quantities,  $p_0$ ,  $p_1$ ,  $p_2$  and  $\delta$  obey the following master equation:

$$\begin{aligned}\frac{dp_0}{dt} &= (\gamma_1 + \mu_1)p_1 + \gamma_2 p_2 - \alpha \delta s_0 p_0 \\ \frac{dp_1}{dt} &= \alpha \delta s_0 p_0 + 2\mu_2 p_2 - (\gamma_1 + \mu_1 + \lambda_1(1 - d_1) + \alpha \delta s_1)p_1 \\ \frac{dp_2}{dt} &= (\alpha \delta s_1 + \lambda_1(1 - d_1))p_1 - (\gamma_2 + 2\mu_2)p_2 \\ \frac{d\delta}{dt} &= \lambda_1 d_1 p_1 + 2\lambda_2 p_2 - (\mu_D + \alpha s_0 p_0 + \alpha s_1 p_1)\delta\end{aligned}\tag{1}$$

or, in matrix form:

$$\frac{dP}{dt} = G(P)P\tag{2}$$

where  $P = [p_0, p_1, p_2, \delta]$  and  $G(P)$  is the following transition matrix function:

$$G(P) = \begin{bmatrix} -\alpha\delta s_0 & \gamma_1 + \mu_1 & \gamma_2 & 0 \\ \alpha\delta s_0 & -\alpha\delta s_1 - \gamma_1 - \mu_1 - \lambda_1(1 - d_1) & 2\mu_2 & 0 \\ 0 & \alpha\delta s_1 + \lambda_1(1 - d_1) & -\gamma_2 - 2\mu_2 & 0 \\ 0 & \lambda_1 d_1 & 2\lambda_2 & -(\alpha(p_0 s_0 + p_1 s_1) + \mu_D) \end{bmatrix} \quad (3)$$

## 154 2.2 Applying the $R_m$ criterion

We want to assess whether the metapopulation is viable, i.e. grows when  
156 individuals are scarce. Intuitively, when individuals are scarce, a good criterion  
for viability is obtained by looking at the “disperser output” yielded by a  
158 unique colonized deme, from its colonization up to its eventual extinction. The  
 $R_m$  criterion (or metapopulation fitness criterion) can be formulated verbally  
160 as (Gyllenberg and Metz, 2001; Metz and Gyllenberg, 2001; Parvinen and  
Metz, 2008):

162 **Criterion 1** *A metapopulation is viable if, and only if, the average number  
of dispersers (emigrants) produced by a population founded by a typical rare  
164 immigrant is greater than 1.*

Here, we develop this intuitive criterion. Suppose that the metapopulation and  
166 the disperser pool are empty. Let us impose an extrinsic flow of immigrants,  
say at rate  $\epsilon$ . Each immigrant has probability  $\frac{\alpha s_0}{\mu_D + \alpha s_0}$  of surviving dispersal  
168 and landing into an empty deme, founding a deme with one individual. Demes  
occupied by one individual are thus created at rate  $\epsilon \frac{\alpha s_0}{\mu_D + \alpha s_0}$ . We want to mon-

itor the flow of dispersers that will be produced by settled individuals, under  
the sole effect of the extrinsic flow of immigrants (the density of dispersers is  
thus kept nil). Let  $z_1$  and  $z_2$  be the probabilities that a deme contains 1 and  
2 individuals, respectively. Equ. 1 now applies to the system at the viability  
boundary:

$$\begin{aligned}\frac{dz_1}{dt} &= 2\mu_2 z_2 - (\gamma_1 + \mu_1 + \lambda_1(1 - d_1))z_1 + \epsilon \frac{\alpha s_0}{\mu_D + \alpha s_0} \\ \frac{dz_2}{dt} &= \lambda_1(1 - d_1)z_1 - (\gamma_2 + 2\mu_2)z_2\end{aligned}\tag{4}$$

Eq. 4 has an invariant measure  $Z(\epsilon)$  given by:

$$\begin{aligned}z_1(\epsilon) &= \epsilon \frac{\alpha s_0}{\mu_D + \alpha s_0} \frac{\gamma_2 + 2\mu_2}{(\gamma_1 + \mu_1)(\gamma_2 + 2\mu_2) + \gamma_2 \lambda_1(1 - d_1)} \\ z_2(\epsilon) &= \epsilon \frac{\alpha s_0}{\mu_D + \alpha s_0} \frac{\lambda_1(1 - d_1)}{(\gamma_1 + \mu_1)(\gamma_2 + 2\mu_2) + \gamma_2 \lambda_1(1 - d_1)} \\ z_0(\epsilon) &= 1 - z_1(\epsilon) - z_2(\epsilon)\end{aligned}\tag{5}$$

The rate at which dispersers will be intrinsically produced from settled indi-  
viduals is thus  $[\lambda_1 d_1, 2\lambda_2] \cdot [z_1(\epsilon), z_2(\epsilon)]$ . For the metapopulation to be viable  
(following our heuristic criterion), this intrinsic rate (emigrant flow) must ex-  
ceed the imposed rate of immigrants (immigrant flow),  $\epsilon$ , i.e.:

$$[\lambda_1 d_1, 2\lambda_2] \cdot [z_1(\epsilon), z_2(\epsilon)] > \epsilon\tag{6}$$

Since the left-hand side of Eq. 6 is linearly dependent on  $\epsilon$  (cf. Eq. 5), the choice  
of  $\epsilon$  has no incidence on the viability property. We can rephrase this criterion  
using symbols from Metz and Gyllenberg (2001). If vector  $A$  is defined as

$$A = [0, \lambda_1 d_1, 2\lambda_2] \quad (7)$$

and vector  $Z$  as  $Z = \frac{Z(\epsilon)}{\epsilon}$ , then the heuristic  $R_m$  viability criterion is

$$R_m = A.Z > 1 \quad (8)$$

184 which develops as:

$$R_m = \frac{s_0 \alpha \lambda_1 (2\lambda_2 (1 - d_1) + d_1 (\gamma_2 + 2\mu_2))}{(s_0 \alpha + \mu_D) (\gamma_2 (\gamma_1 + \mu_1 + \lambda_1 (1 - d_1)) + 2\mu_2 (\gamma_1 + \mu_1))} \quad (9)$$

Vector  $Z$  actually describes the “typical rarity” of the species, i.e. it contains  
186 the weights that should be given to every patch state. These weights are  
analogous to the “ $l_x$ ” coefficient used in stage-structured populations (Caswell,  
188 2001), i.e. the proportion of individuals that survive from birth to stage  $x$ .  
Here,  $z_x$  yields the relative weight of state  $x$  in the computation of the  $R_m$ ,  
190 i.e. a measure of the total time passed in state  $x$  in the course of a patch  
lifetime.

192 If  $Y$  is the probabilistic state of a patch that has just received a rare immigrant  
(in our present example  $Y = [0, 1, 0]$ ), then  $Z = -G_0^{-1}.Y$ , with  $G_0$  being  $G$   
194 computed without the immigration pressure of the rare species. In terms of  
dimensions,  $Y$  is dimensionless and  $G$  is a matrix of (typically negative) rates,  
so that  $G_0^{-1}$  and  $Z$  are times.  $R_m$  is dimensionless since it is the product of  
196 vector  $A$  (a vector of rates, i.e. inverse of times) and vector  $Z$  (eq. 8).

### 198 2.3 Proving the criterion mathematically

The criterion given in section 2.2 is formulated in biological terms: it addresses  
200 the problem of metapopulation viability using a quantity (the “lifetime pro-  
duction of emigrants” of a patch) with a tangible ecological meaning. The  
202 intuitive explanation of this criterion has already been given by Metz and  
Gyllenberg (2001). This does not constitute a proof of its validity, and the  
204 simple formulation retained may not be suitable for the whole class of related  
metapopulation models.

206 Assessing whether a metapopulation described by equation 1 is viable is equiv-  
alent to assessing whether the linearized dynamical system implied by equa-  
208 tion 1, when the metapopulation is almost empty, is unstable. Instability may  
be looked for by studying the eigenvalues of the linearized system: when all  
210 eigenvalues have negative real parts, the system is stable; when at least one  
eigenvalue has positive real part, the system is unstable. The  $R_m$  criterion  
212 should thus be shown to summarize the behavior of the dominant eigenvalue  
of the linearized system, using a single number.

214 That’s what we establish in the next section, providing a more general formu-  
lation of the  $R_m$  criterion, and translating it into biological terms.

### 216 **3 Mathematical proof of the criterion: a digest**

The complete proof of the  $R_m$  criterion can be found in Appendix A. The Ap-  
218 pendix is self-sufficient, so that interested readers may read it without coming  
back incessantly to the main text. Here, we only give a sketch of the proof,  
220 highlighting its key issues and assumptions.

#### *3.1 General model and assumptions*

222 We consider a closed spatially implicit metapopulation occupied by individ-  
uals from one species. We note  $P$  the vector that describes the state of the  
224 metapopulation, i.e. given first by all the probabilities that a sample patch is  
found containing 0, 1, 2,..., $N$  individuals (possibly structured by classes, sex,  
226 etc.), and then by all densities of “free” individuals that do not reside in a  
patch (e.g. the density of dispersers, if the model uses a disperser pool as in  
228 the toy model above). The size of vector  $P$  is at least  $N + 1$ .

We note  $G(P)$  the “transition matrix”, i.e. the matrix function that determines  
the master equation of  $P$ :

$$\frac{dP}{dt} = G(P)P \quad (10)$$

Elements of  $G$  correspond to transition rates, i.e. the rates at which popula-  
230 tions change in state (upper diagonal block of  $G$ ), the rates of free individual  
production (lower left-hand block), and the mortality and flows among cate-  
232 gories of free individuals (lower diagonal block).

More specifically, the first column of  $G$  ( $g_{.,1}$ ) represents the rates (i) of tran-  
234 sition from an empty population to a population containing at least one indi-

vidual, and (ii) of free individual production due to empty populations (that  
236 are necessarily 0).

In the simplest models, only  $g_{2,1}$  is non null (i.e. the only possible transition for  
238 an empty patch is to be colonized by one individual). More complicated models  
can be envisaged in which individuals belong to different classes (e.g. wingless  
240 and winged individuals in aphids, or males and hermaphrodites, see our third  
Example), or in which dispersal occurs in groups of more than one individual  
242 (e.g. fruits with multiple seeds). In these cases, more than one component of  
 $g_{.,1}$  may be positive. Positive components of  $g_{.,1}$  are called *colonization rates*,  
244 and are functions of  $P$ . We assume that there are  $q$  such colonization rates,  
noted  $(M_k)_{k \in [1;q]}$ .

246 We are interested in determining whether eq. 1 displays an unstable equilib-  
rium at  $P^* = [1, 0, 0, \dots, 0]$ . This is achieved through the knowledge of the  
248 eigenvalues of equation 10, linearized near  $P^*$ .

### 3.2 Linearization of the equation, and definition of the initial states

250 Colonization rates  $M_k$  are assumed to obey two simple rules (see Appendix  
A):

- 252 (1) colonization rates are not directly affected by the proportion of empty  
patches (i.e.  $\partial M_k / \partial p_0 = 0$  for all  $k$ );
- 254 (2) when the metapopulation is almost empty, colonization rates increase  
with an increase of any population state probability (except empty pop-  
256 ulation state) and any density of free individuals (i.e. for all  $k$ , the vector  
 $\partial_P M_k$  taken at  $P = P^*$  has only non-negative components).

These two conditions allow the following transformation of equation 10 near the equilibrium  $P^*$  (section A.3):

$$\frac{dP}{dt} \approx G(P^*) \cdot P + \sum_{k=1}^q (A_k \cdot P) Y_k \quad (11)$$

where the vectors  $Y_k$  correspond to the different possible initial states of recently colonized patches (formally,  $Y_k = \partial_{M_k} g_{.,1}$ ), and the vectors  $A_k$  are the first-order production rates of emigrants that will found populations with initial state  $Y_k$  (formally,  $A_k = \partial_P M_k$ ).

Following equation 11, the viability problem is equivalent to finding the dominant eigenvalue of matrix  $G + \sum Y_k A_k^T$  at the equilibrium  $P^*$ . The eigenvalue corresponding to its first row and first column is always 0 (the empty state is neutrally stable), which simplifies the issue.

From now on, tilde quantities ( $\widetilde{X}$  and so forth) indicate vectors and matrices whose first row/column have been removed. The final eigenvalue problem is reduced to finding the dominant eigenvalue of matrix  $\widetilde{J}$  defined by:

$$\widetilde{J} = \widetilde{G} + \sum_{k=1}^q \widetilde{Y}_k \widetilde{A}_k^T \quad (12)$$

(for clarity, the dependence of  $G$  on  $P^*$  will now be omitted).

### 3.3 Simplification of the eigenvalue calculation

Finding the dominant eigenvalue of  $\widetilde{J}$  is made easier by following two simplification steps. First, it can be proved that matrix  $\widetilde{J}$  admits one real dominant eigenvalue (section A.4).

Second, some manipulations (sections A.6 and A.7) show that if we consider



the  $q \times q$  matrix  $R(\lambda)$ , defined by its elements  $r_{ij}(\lambda)$ :

$$r_{ij}(\lambda) = -\widetilde{A}_i \cdot (\widetilde{G} - \lambda I)^{-1} \widetilde{Y}_j \quad (13)$$

then the problem of finding the greatest real eigenvalue of  $\widetilde{J}$ ,  $\lambda$ , is turned into  
272 finding a matrix  $R(\lambda)$  that admits 1 as an eigenvalue.

Matrix  $R(0)$  contains the flows of migrant individuals produced, weighted by  
274 population lifetimes. Indeed,  $\widetilde{Y}_j$  is the  $j^{th}$  possible initial state of a newly colo-  
nized patch.  $-\widetilde{G}^{-1}\widetilde{Y}_j$  corresponds to the quasi-equilibrium state probabilities  
276 for a population that began in state  $\widetilde{Y}_j$  (cf. the  $Z$  vector of section 2). On  
average, a population that began in state  $\widetilde{Y}_j$  spends a proportion  $[-\widetilde{G}^{-1}\widetilde{Y}_j]_k$   
278 of its lifetime (i.e. up to its eventual extinction) in state  $k$ .

The product  $-\widetilde{A}_i \cdot \widetilde{G}^{-1}\widetilde{Y}_j$  thus gives the amount of emigrants that will found  
280 new populations starting in state  $\widetilde{Y}_i$ , produced by populations that initially  
started in state  $\widetilde{Y}_j$ . For some positive  $\lambda$ , the matrix  $R(\lambda)$  corresponds to the  
282 same quantities, but with an artificial increase in the extinction rates of the  
populations, quantified by  $\lambda$ .

### 284 3.4 Final steps and the $R_m$ criterion

The  $R_m$  criterion is found by focusing on the dominant eigenvalue  $m(\lambda)$  of  $R(\lambda)$   
286 (see sections A.7 and A.8). The dominant eigenvalue of  $R$  is a non-negative  
continuous decreasing functions of  $\lambda$ , that converges towards 0 (since all  $r_{ij}$   
288 have these properties). Therefore, by checking that the dominant eigenvalue  
of  $R(0)$  is strictly greater than 1, one effectively checks that there exists some  
290 positive  $\lambda$  such that the dominant eigenvalue of  $R(\lambda)$  is 1. Conversely, if such  
a  $\lambda$  exists, the dominant eigenvalue of  $R(0)$  must be greater than 1, given that

292 it is a decreasing function of  $\lambda$ .

The mathematical formulation of the  $R_m$  criterion is thus:

$$m(0) > 1 \Leftrightarrow \text{Sp}(\tilde{J}) \cap \mathbb{R}_+^* \neq \emptyset \quad (14)$$

### 3.5 *Biological interpretation of the criterion*

294 Interpreting the  $R_m$  criterion in biological terms requires understanding the  
signification of  $m(0)$ . As already stated, the matrix  $R(0)$  contains the flows of  
296 emigrant productions weighted by population lifetimes.

Each column of this matrix corresponds to a colonization pathway, i.e. an  
298 initial state  $Y_k$ . The life cycles of patches are thus distinguished in different  
classes, depending on the initial state in which they began. Each class of life  
300 cycle initiates other classes of life cycles (the rows of  $R(0)$ ) throughout its  
life.  $R(0)$  can thus be seen as a transition matrix between classes of patch life  
302 cycles, just as a Lefkovitch matrix is a transition matrix between classes of  
individuals.

304 The dominant eigenvalue of this matrix,  $m(0)$ , can be interpreted as the  
“metapopulation basic reproduction ratio”, by analogy with the  $R_0$  criterion,  
306 the basic reproduction ratio of stage-structured populations. If greater than  
one, the population of patches initially grows (viability), otherwise it will de-  
308 cline.

When newly colonized patches have only one initial state ( $\tilde{Y}$ , with the corre-  
310 sponding emigrant production vector,  $\tilde{A}$ ),  $R$  reduces to a  $1 \times 1$  matrix with  
only one eigenvalue equal to  $-\tilde{A}.\tilde{G}^{-1}\tilde{Y}$  (the expression given by Metz and

312 Gyllenberg, 2001), just like a matrix population model reduces to a scalar  
population model when individuals are not structured. When there are sev-  
314 eral population initial states, the computation of the  $R_m$  criterion becomes  
more complicated, paralleling the complexification of  $R_0$  in stage-structured  
316 populations (Diekmann et al., 1990).

## 4 A recipe for the analytical computation of $R_m$ , with examples

### 318 4.1 The recipe

Based on our proof of the  $R_m$  criterion, we give a simple five-point recipe that  
320 helps make no mistake in computing the  $R_m$  value for a theoretical metapop-  
ulation model:

- 322 (1) clearly state the transition matrix  $G(P)$ , extract its first column  $g_{.,1}$  when  
 $P = P^*$ , and find the colonization rates  $M_k$ ;
- 324 (2) find the corresponding initial population states  $Y_k = \partial_{M_k} g_{.,1}$ ;
- (3) compute the emigrant production vectors,  $A_k = \partial_P M_k$ ;
- 326 (4) compute the inverse of the reduced transition matrix,  $\tilde{G}^{-1}$ ;
- (5) collect the  $r_{ij}(0) = -\tilde{A}_i \cdot \tilde{G}^{-1} \tilde{Y}_j$  in matrix  $R(0)$  (tilde versions of matrices  
328 and vectors are obtained by removing the first row/column) and compute  
 $R_m = \rho[R(0)]$ , the dominant eigenvalue of  $R(0)$ .

330 Obviously this recipe is only efficient to get analytical expressions for  $R_m$ . For  
numerical calculations, it is probably better to use an approximated procedure,  
332 such as the one provided by Metz and Gyllenberg (2001), instead of computing  
the inverse of the reduced transition matrix.

334 We now illustrate our recipe with three examples. We first provide two exam-  
ples in which the  $R_m$  criterion is very simple, and then one example in which  
336 the  $R_m$  criterion is more complicated to compute. All these models are sim-  
ple extensions of Metz and Gyllenberg's model presented in section 2.1, and  
338 they retain the same notations and deme size  $N = 2$ . These models have not  
been explicitly designed to answer biological questions (even if they convey  
340 some interesting results, assumptions of these models are quite unrealistic),  
but rather to illustrate how to apply the  $R_m$  fitness concept. Each example is  
342 presented briefly, then the recipe given in this section is applied to the model,  
and finally we perform a simplification of the example to get some tractable  
344 results.

## 4.2 *Example 1: an animal model with mobile adults*

### 346 4.2.1 *The model*

Consider a metapopulation as in section 2.1, but in which all individuals  
348 (adults and juveniles) can disperse (e.g. to escape crowded demes) and enter  
the disperser pool. Assume for simplicity that after dispersal, adults behave  
350 just as juveniles migrants do. As before, individuals that were not accepted  
in a deme get back to the disperser pool, hence the qualifier of active dis-  
352 perser pool. We slightly change the parameterization and state that  $e_i$  is the  
per capita rate of emigration from a deme with  $i$  individuals. The model is  
354 summarized in Fig. 2.

#### 4.2.2 Step 1: transition matrix and colonization rates

356 The  $G$  transition matrix (when  $N = 2$ ) is rewritten as (cf. Casagrandi and Gatto, 2002; Parvinen et al., 2003):

$$G(P) = \begin{bmatrix} -\alpha\delta s_0 & \gamma_1 + \mu_1 + e_1 & \gamma_2 & 0 \\ \alpha\delta s_0 & -\alpha\delta s_1 - \gamma_1 - \mu_1 - \lambda - e_1 & 2(\mu_2 + e_2) & 0 \\ 0 & \alpha\delta s_1 + \lambda & -\gamma_2 - 2(\mu_2 + e_2) & 0 \\ 0 & d_1 & 2d_2 & -(\alpha \langle s_i \rangle + \mu_D) \end{bmatrix} \quad (15)$$

In this example, the first column of  $G$  only depends on one colonization rate:

$$M = \alpha s_0 \delta \quad (16)$$

#### 358 4.2.3 Step 2: initial population states

The initial state vector  $Y = \partial_M g_{.,1}$  is given by:

$$Y = [-1, 1, 0, 0] \quad (17)$$

#### 4.2.4 Step 3: emigrant production vectors

The emigrant production vector  $A = \partial_P M$  is given by:

$$A = [0, 0, 0, \alpha s_0] \quad (18)$$

#### 360 4.2.5 Step 4: inverse of the transition matrix

The reduced transition matrix is:

$$\tilde{G} = \begin{bmatrix} -\gamma_1 - \mu_1 - \lambda - e_1 & 2(\mu_2 + e_2) & 0 \\ \lambda & -\gamma_2 - 2(\mu_2 + e_2) & 0 \\ e_1 & 2e_2 & -(\alpha s_0 + \mu_D) \end{bmatrix} \quad (19)$$

Its inverse is:

$$\tilde{G}^{-1} = \frac{1}{\det \tilde{G}} \begin{bmatrix} a_2(\alpha s_0 + \mu_D) & 2(\mu_2 + e_2)(\alpha s_0 + \mu_D) & 0 \\ \lambda(\alpha s_0 + \mu_D) & a_1(\alpha s_0 + \mu_D) & 0 \\ a_2 e_1 + 2\lambda e_2 & 2\mu_2 e_1 + 2(a_1 + e_1)e_2 & 2(\mu_2 + e_2)(\mu_1 + \gamma_1 + e_1) + \gamma_2 a_1 \end{bmatrix} \quad (20)$$

where

$$a_1 = \gamma_1 + \mu_1 + \lambda + e_1 \quad (21)$$

$$a_2 = \gamma_2 + 2(\mu_2 + e_2) \quad (22)$$

$$\det \tilde{G} = -[2(\mu_2 + e_2)(\mu_1 + \gamma_1 + e_1) + \gamma_2 a_1](\alpha s_0 + \mu_D) \quad (23)$$

#### 4.2.6 Step 5: $R$ matrix and $R_m$ calculation

362 The value of  $R_m$  is obtained from the four previous steps:

$$R_m = \frac{s_0 \alpha (2e_2(e_1 + \lambda) + e_1(\gamma_2 + 2\mu_2))}{(s_0 \alpha + \mu_D)(\gamma_2 \lambda + (\gamma_1 + \mu_1 + e_1)(\gamma_2 + 2(e_2 + \mu_2)))} \quad (24)$$

364 The condition  $R_m = 1$  in the three-parameter space  $(e_1, e_2, \lambda)$  is represented in Fig. 3. This condition is a surface which separates the region of viability (above the surface) from the region of extinction (under the surface). What

we observe is that viability is easier when  $e_1$  is low,  $\lambda$  is high and  $e_2$  is not too low (Fig. 3).

#### 4.2.7 Application

We note  $c = \frac{\mu_D}{\alpha s_0 + \mu_D}$  the cost of dispersal. When the species is “environmentally blind” (i.e.  $e$  cannot be adjusted according to local population size) and parameters are locally population size-independent (i.e.  $\mu_1 = \mu_2 = \mu$  and  $\gamma_1 = \gamma_2 = \gamma$ ), we obtain:

$$R_m = \frac{(1-c)[2e(e+\lambda) + e(\gamma+2\mu)]}{\gamma\lambda + (\gamma+\mu+e)[\gamma+2(e+\mu)]} \quad (25)$$

so that the condition  $R_m > 1$  requires:

- (1)  $\lambda > \mu + \gamma$ , i.e. births more than compensate deaths and extinctions;
- (2) a low value of  $c$ , i.e. the reward of dispersing to other patches is not too low (the maximal value of  $c$  is a complicated function of parameters; when  $\gamma \rightarrow 0$ , this can be approximated by  $c < \left(\frac{\lambda-\mu}{\lambda+\mu}\right)^2$ );
- (3) an intermediate value of  $e$  (the maximal and minimal values of  $e$  are complicated functions, but when  $\gamma \rightarrow 0$ , this simplifies to  $\frac{\lambda-\mu-c(\lambda+\mu)-\sqrt{(1-c)[(\lambda-\mu)^2-c(\lambda+\mu)^2]}}{2c} < e < \frac{\lambda-\mu-c(\lambda+\mu)+\sqrt{(1-c)[(\lambda-\mu)^2-c(\lambda+\mu)^2]}}{2c}$ ). This condition is intuitive since low  $e$  prevents the rescue of extinct patches (due to stochastic deaths or extinction) by occupied patches, and high  $e$  diminishes the average occupancy of patches, and thus lowers the metapopulation reproduction rate.

We can visually check these analytical conditions of viability by looking at the intersection of the plane  $e_1 = e_2 = e$  with the surface depicted in Fig. 3. Viability is easiest to achieve when  $\lambda$  is high and  $e$  is intermediate (with the parameters of Fig. 3, this occurs for  $e \approx 1.7$ ).

### 384 4.3 Example 2: a plant model with passive instantaneous dispersal

#### 4.3.1 The model

Consider now a metapopulation of plants. In this case, adults are sessile (as in section 2.1) and dispersal is passive (propagules that fail to settle in a deme just die, and do not get back to the dispersal pool: there is no “second chance”, as in seed bank models). A potential colonizer arriving in a deme with  $i$  individuals has probability  $1 - s_i$  to die. Let us assume that propagules immediately try to settle in a deme and have vanishingly short lifespan (you may think of vegetative propagules with little nutritive and protective tissues). This amounts to saying that the dynamics of the disperser pool is very fast compared to the dynamics of demes, so that the disperser pool always attains its equilibrium. In mathematical terms:

$$\frac{d\delta}{dt} = \sum_{k=0}^N k\lambda_k d_k p_k - (\alpha + \mu_D)\delta = 0 \quad (26)$$

which yields:

$$(\alpha + \mu_D)\delta = \langle i\lambda_i d_i \rangle \quad (27)$$

with  $\langle i\lambda_i d_i \rangle = \lambda_1 d_1 p_1 + 2\lambda_2 p_2$  when  $N = 2$ . We note  $c = \frac{\mu_D}{\alpha + \mu_D}$  the potential cost of dispersal. The realized cost of dispersal  $c'$  (measuring the loss in probability of settlement of a propagule relative to local reproduction) is given as the probability of dying during dispersal, plus the probability of surviving dispersal but not being able to settle in a patch, i.e.:

$$c' = c + (1 - c)(1 - \langle s_i \rangle) \quad (28)$$

386 The realized cost of dispersal depends on the state of the metapopulation since



there is a greater cost associated to dispersal in a more crowded metapopu-  
388 lation. The model is summarized in Fig. 4. We need not bother about the  $\delta$   
column and row of the transition matrix any more since immigration pressures  
390 can be directly expressed as functions of  $\langle i\lambda_i d_i \rangle$  (cf. equation 27).

#### 4.3.2 Step 1: transition matrix and colonization rates

The  $G$  transition matrix (when  $N = 2$ ) is rewritten as (cf. Casagrandi and  
Gatto, 2002; Parvinen et al., 2003):

$$G(P) = \begin{bmatrix} -\langle i\lambda_i d_i \rangle s_0(1-c) & \gamma_1 + \mu_1 & \gamma_2 \\ \langle i\lambda_i d_i \rangle s_0(1-c) & -\langle i\lambda_i d_i \rangle s_1(1-c) - \gamma_1 - \mu_1 - \lambda_1(1-d_1) & 2\mu_2 \\ 0 & \langle i\lambda_i d_i \rangle s_1(1-c) + \lambda_1(1-d_1) & -\gamma_2 - 2\mu_2 \end{bmatrix} \quad (29)$$

In this example, the first column of  $G$  only depends on one colonization rate:

$$M = s_0(1-c) \langle i\lambda_i d_i \rangle \quad (30)$$

#### 392 4.3.3 Step 2: initial population states

The initial state vector  $Y = \partial_M g_{.,1}$  is given by:

$$Y = [-1, 1, 0] \quad (31)$$

#### 4.3.4 Step 3: emigrant production vectors

The emigrant production vector  $A = \partial_P M$  is given by:

$$A = s_0(1-c)[0, \lambda_1 d_1, 2\lambda_2] \quad (32)$$

#### 394 4.3.5 Step 4: inverse of the transition matrix

The reduced transition matrix is:

$$\tilde{G} = \begin{bmatrix} -\gamma_1 - \mu_1 - \lambda_1(1 - d_1) & 2\mu_2 \\ \lambda_1(1 - d_1) & -\gamma_2 - 2\mu_2 \end{bmatrix} \quad (33)$$

Its inverse is:

$$\tilde{G}^{-1} = \frac{1}{\det \tilde{G}} \begin{bmatrix} -\gamma_2 - 2\mu_2 & -2\mu_2 \\ -\lambda_1(1 - d_1) & -\gamma_1 - \mu_1 - \lambda_1(1 - d_1) \end{bmatrix} \quad (34)$$

where

$$\det \tilde{G} = [\gamma_1 + \mu_1 + \lambda_1(1 - d_1)] [\gamma_2 + 2\mu_2] - 2\mu_2\lambda_1(1 - d_1) \quad (35)$$

#### 4.3.6 Step 5: $R$ matrix and $R_m$ calculation

The value of  $R_m$  is obtained from the four previous steps:

$$R_m = \frac{s_0(1 - c)\lambda_1(2\lambda_2(1 - d_1) + d_1(\gamma_2 + 2\mu_2))}{\gamma_2(\gamma_1 + \mu_1 + \lambda_1(1 - d_1)) + 2\mu_2(\gamma_1 + \mu_1)} \quad (36)$$

396 Let us compare this result to the expression of  $R_m$  that we found for the open-  
model of section 2.1 (Eq. 9). Two major modifications were introduced in  
398 the present model: passive colonization and instantaneous dispersal. Passive  
colonization introduces a difference in the way dispersers that failed to set-  
400 tle are treated: they die instead of getting back to the disperser pool. This  
basically changes the cost of dispersal. With active colonization, the physi-  
402 ological cost of dispersal would be  $c(P) = \frac{\mu_D}{\alpha\langle s_i \rangle + \mu_D}$ . Unsurprisingly, it would  
be reduced (as dispersers have “second chances”) and it would depend on the

404 metapopulation state (as dispersers are more likely to leave the disperser pool  
in less crowded metapopulations). The value of this cost in a nearly empty  
406 metapopulation (as is relevant for viability analysis) would be  $c^* = \frac{\mu_D}{\alpha s_0 + \mu_D}$ . If  
we replace  $s_0(1-c)$  by  $(1-c^*)$  in Eq. 36, we exactly recover the  $R_m$  expression  
408 of section 2.1 (see Eq. 9). This means that the second modification (instant-  
aneous dispersal) did not have any effect, except on the experienced cost of  
410 dispersal. It is remarkable that taking into account the delay due to dispersal  
has no effect on the formulation of the  $R_m$  criterion.

#### 412 4.3.7 Application

When parameters are locally population size-independent (i.e.  $\lambda_1 = \lambda_2 = \lambda$ ,  
 $\mu_1 = \mu_2 = \mu$  and  $\gamma_1 = \gamma_2 = \gamma$ ), we obtain (with the notation  $d = d_1$ ):

$$R_m = \frac{s_0(1-c)\lambda [2\lambda(1-d) + d(\gamma + 2\mu)]}{\gamma [\gamma + \mu + \lambda(1-d)] + 2\mu(\gamma + \mu)} \quad (37)$$

The condition  $R_m > 1$  thus requires:

414 (1)  $\lambda > \gamma + \mu$  (as in the previous example);

(2) one of the three following conditions:

416 (a)  $c < 1 - \frac{\mu+\gamma}{\lambda}$ ,  $\frac{(\gamma+2\mu)(\gamma+\mu+\lambda)-2\lambda\mu}{2\lambda^2(1-c)} < s < \frac{\mu+\gamma}{\lambda(1-c)}$  and  $d < \frac{\gamma(\lambda+3\mu+\gamma)+2\mu^2-2(1-c)s\lambda^2}{\lambda[(1-c)s[2(\lambda-\mu)-\gamma]-\gamma]}$ ;

(b)  $c < 1 - \frac{\mu+\gamma}{\lambda}$  and  $s > \frac{\mu+\gamma}{\lambda(1-c)}$ ;

418 (c)  $1 - \frac{\mu+\gamma}{\lambda} < c < \frac{[\lambda-\mu-\gamma][\gamma+2(\mu+\lambda)]}{2\lambda^2}$ ,  $s > \frac{(\gamma+2\mu)(\gamma+\mu+\lambda)-2\lambda\mu}{2\lambda^2(1-c)}$  and  $d < \frac{\gamma(\lambda+3\mu+\gamma)+2\mu^2-2(1-c)s\lambda^2}{\lambda[(1-c)s[2(\lambda-\mu)-\gamma]-\gamma]}$ .

420 Conditions on  $c$ ,  $s$  and  $d$  are intuitive: either the dispersal cost is low and the  
acceptance rate is high (case b), or dispersal must be limited to counteract  
422 the effect of a low acceptance rate (case a) or of a high dispersal cost (case  
c). A strictly positive  $d$  is not mandatory since completely occupied patches

424 provide emigrants anyway (i.e.  $d_2 = 1$ ). When dispersal cost is too high, or  
both acceptance rate is low and dispersal cost is high, viability is not possible,  
426 whatever the value of  $d$ .

#### 4.4 *Example 3: a model with hermaphrodites and males*

##### 428 4.4.1 *The model*

We now elaborate a more complex model based on the example given in section  
430 2.1. We assume, as before, that each deme contains from 0 to 2 individuals,  
and we consider parameters describing the emigration rates of adults ( $e$ , as  
432 in section 4.2) and the dispersal probabilities of newborn offspring ( $d$ , as in  
section 4.3). We consider that the metapopulation is made up of two classes  
434 of individuals: hermaphrodites (all parameters given by index  $H$ ) and males  
(all parameters given by index  $M$ ).

436 There are two mortality rates ( $\mu_H$  and  $\mu_M$ ), two emigration rates ( $e_H$  and  $e_M$ )  
and two dispersal probabilities for offspring born in not fully occupied demes  
438 ( $d_H$  and  $d_M$ ; offspring born in fully occupied demes always disperse) corre-  
sponding to these two classes. All migration pressures are computed directly  
440 (i.e. there is no density of dispersers, like in section 4.3). These assumptions  
lead to 6 state probabilities, noted as: state 0 (empty patch), state 1 (one  
442 hermaphrodite), state 2 (2 hermaphrodites), state 3 (one male), state 4 (two  
males), and state 5 (one hermaphrodite and one male).

In this model, hermaphrodite individuals can reproduce in two ways: they  
either self-fertilize (always the case in state 1, and it happens with proba-  
bility  $\sigma$  in states 2 and 5), or outcross with another individual (probability

$1 - \sigma$  in states 2 and 5). Self-fertilized offspring suffers from inbreeding depression, and a fraction  $D$  of these offspring die. The per capita fecundities of hermaphrodites in states 1, 2 and 5 are thus modelled as ( $\lambda$  is the base fecundity):

$$\lambda_1 = (1 - D)\lambda \quad (38)$$

$$\lambda_2 = \lambda_5 = (1 - \sigma D)\lambda \quad (39)$$

444 At birth, a proportion  $f$  of offspring are hermaphrodites. Catastrophes happen  
at a rate  $\gamma$ , the cost of dispersal is  $c$ , and the two colonization rates are noted  
446  $M_H$  and  $M_M$ .

#### 4.4.2 Step 1: transition matrix and colonization rates

The transition matrix is quite cumbersome. Instead we give its (transposed) first column:

$$g_{.,1}^T = [-(1 - c)s_0(m_H + m_M), (1 - c)s_0m_H, 0, (1 - c)s_0m_M, 0, 0] \quad (40)$$

with  $m_H$  and  $m_M$  the raw immigration pressures defined by:

$$m_H = e_H(p_1 + 2p_2 + p_5) + f[\lambda_1 d_H p_1 + 2\lambda_2 p_2 + \lambda_5 p_5] \quad (41)$$

$$m_M = e_M(p_3 + 2p_4 + p_5) + (1 - f)[\lambda_1 d_M p_1 + 2\lambda_2 p_2 + \lambda_5 p_5] \quad (42)$$

$m_H$  and  $m_M$  account for (i) mobile adults that emigrate (first term) and (ii) newborns that disperse (second term). The colonization rates  $M_H$  and  $M_M$  are given by:

$$M_H = (1 - c)s_0m_H \quad (43)$$

$$M_M = (1 - c)s_0m_M \quad (44)$$

After accounting for the effect of inbreeding depression (equations 38 and 39),  
colonization rates are:

$$M_H = (1-c)s_0 \{e_H(p_1 + 2p_2 + p_5) + \lambda f [(1-D)d_H p_1 + (1-\sigma D)(2p_2 + p_5)]\} \quad (45)$$

$$M_M = (1-c)s_0 \{e_M(p_3 + 2p_4 + p_5) + \lambda(1-f) [(1-D)d_M p_1 + (1-\sigma D)(2p_2 + p_5)]\} \quad (46)$$

#### 448 4.4.3 Step 2: initial population states

Reduced initial population states,  $\widetilde{Y}_H$  and  $\widetilde{Y}_M$ , are obtained using equation  
40:

$$\widetilde{Y}_H = [1, 0, 0, 0, 0] \quad (47)$$

$$\widetilde{Y}_M = [0, 0, 1, 0, 0] \quad (48)$$

i.e. rare hermaphrodites begin in state 1 and rare males in state 3. Note that  
450 the first element of the non-reduced  $Y$  vectors is always  $-1$ , but we directly  
presented reduced vectors in this example.

#### 452 4.4.4 Step 3: emigrant production vectors

Emigrant production rates are given by equations 45 and 46:

$$\widetilde{A}_H = (1-c)s_0[e_H + \lambda f(1-D)d_H, 2e_H + 2\lambda f(1-\sigma D), 0, 0, e_H + \lambda f(1-\sigma D)] \quad (49)$$

$$\widetilde{A}_M = (1-c)s_0[\lambda(1-f)(1-D)d_M, 2\lambda(1-f)(1-\sigma D), e_M, 2e_M, e_M + \lambda(1-f)(1-\sigma D)] \quad (50)$$

#### 4.4.5 Step 4: inverse of the transition matrix

The calculation of  $\tilde{G}^{-1}$  is tedious and its writing uneasy, so we will stick here to the expressions of  $Z_H$  and  $Z_M$ . The quasi-equilibrium distribution of states after initial hermaphrodite colonization is given by:

$$Z_H = \frac{\left[ (\gamma + 2a_H)(\gamma + a_H + a_M), c_H(\gamma + a_H + a_M), \frac{c_M a_H (\gamma + 2a_H)}{\gamma + a_M}, 0, c_M(\gamma + 2a_H) \right]}{(\gamma + a_H + a_M) [(\gamma + a_H + b)(\gamma + 2a_H) - 2c_H a_H] - c_M a_M (\gamma + 2a_H)} \quad (51)$$

where

$$a_H = e_H + \mu_H \quad (52)$$

$$a_M = e_M + \mu_M \quad (53)$$

$$b = (1 - D)(1 - (1 - f)d_M - d_H f)\lambda \quad (54)$$

$$c_H = (1 - D)(1 - d_H)\lambda f \quad (55)$$

$$c_M = (1 - D)(1 - d_M)(1 - f)\lambda \quad (56)$$

454 The only component of  $Z_H$  equal to 0 is the one corresponding to state 4 (two  
males in a patch), as this state can only be reached through the immigration  
456 of a male in a patch already occupied by one male (state 3) and hence cannot  
be attained in the absence of immigrants.

The quasi-equilibrium distribution of states after initial male colonization is given by:

$$Z_M = [0, 0, \frac{1}{\gamma + e_M + \mu_M}, 0, 0] \quad (57)$$

458 As males do not reproduce on their own, they either die in the patch they  
found or emigrate towards some other patch. Thus, the average time passed  
460 by a male in a patch (state 3) is the inverse of the sum of death, catastrophe  
and emigration rates.

#### 4.4.6 Step 5: $R$ matrix and $R_m$ calculation

The term  $r_{HM} = \widetilde{A}_H \cdot Z_M$  equals 0, so that eigenvalues of  $R$  are easily obtained (they are the diagonal elements of the matrix). The eigenvalue associated with the male patch cycle is:

$$r_{MM} = \frac{(1-c)s_0 e_M}{\gamma + e_M + \mu_M} \quad (58)$$

which is always inferior to 1. The eigenvalue associated with the hermaphrodite patch cycle ( $r_{HH}$ ) yields the expression of  $R_m$ :

$$R_m = \frac{(1-c)s_0 \{(\gamma + 2a_H) [(e_H + (1-D)d_H f \lambda)a_H + c_M(e_H + x)] + 2c_H(\gamma + a_H + a_M)(e_H + x)\}}{(\gamma + a_H + a_M) [(\gamma + a_H + b)(\gamma + 2a_H) - 2c_H a_H] - c_M a_M (\gamma + 2a_H)} \quad (59)$$

where

$$x = \lambda f(1 - \sigma D) \quad (60)$$

#### 4.4.7 Application

The formula for  $R_m$  is quite complicated to analyze in detail. In Fig. 5, regions of viability are presented using projections on two-parameter planes. The different panels of Fig. 5 suggest the existence of constraints on the covariations of  $c$  and  $e$  (for instance). These constraints seem rather strong (except perhaps for constraints due to inbreeding depression on  $d$ ,  $e$ , and  $\sigma$ ).

We can remove the dimension of  $\lambda$  from the expression of  $R_m$ . Let  $\zeta = \gamma/\lambda$ ,  $q = b/\lambda$ ,  $y = x/\lambda$ ,  $f_H = e_H/\lambda$ ,  $k_H = a_H/\lambda$ ,  $k_M = a_M/\lambda$ ,  $l_H = c_H/\lambda$ , and  $l_M = c_M/\lambda$ . The new expression for  $R_m$  is independent of  $\lambda$  and given by:

$$R_m = \frac{(1-c)s_0 \{(\zeta + 2k_H) [(f_H + (1-D)d_H f)k_H + l_M(f_H + y)] + 2l_H(\zeta + k_H + k_M)(f_H + y)\}}{(\zeta + k_H + k_M) [(\zeta + k_H + q)(\zeta + 2k_H) - 2l_H k_H] - l_M k_M (\zeta + 2k_H)} \quad (61)$$

At the limit  $f \approx 1$ , the value of  $R_m$  can be computed as the sum of the pure



hermaphrodite fitness,  $R_m^H$  and an infinitesimal term  $(1-f)r_M$  that represents the contribution of males to the metapopulation fitness.

$$R_m^H = \frac{(1-c)s_0 \{f_H [\zeta + 2(1-D+k_H)] + 2l_H y + d_H(1-D)(\zeta + 2(k_H - f_H))\}}{(\zeta + 2k_H)(\zeta + k_H) + l_H \zeta} \quad (62)$$

## 5 From viability to invasibility

### 5.1 Add-ons to the recipe

The  $R_m$  criterion formulated so far is valid only to infer the viability of a metapopulation, i.e. to assess whether a single-species almost empty metapopulation can grow. A germane problem is the invasion of a species in a metacommunity. A species can invade a metacommunity if it can grow in abundance when it is very rare. The proof given in appendix A is also true for the invasibility problem (the problem only needs to be rewritten in a more general fashion, but the core arguments hold).

For invasion problems, the methodology we described has to be adapted as follows:

- (1) for invasibility problems, the  $G$  matrix is much larger: there is a patch state for each combination of individuals from each species (in general, if the number of potential species is  $S$  and each patch can host at most  $N$  individuals, there are  $\binom{N+S}{N} = \frac{(N+S)!}{N!S!}$  patch states; when  $S \gg N$ , the number of patch states is equivalent to  $\frac{S^N}{N!}$  as  $S \rightarrow \infty$ ). For instance, if only 2 species are present and the maximum number of individuals per patch is fixed at 3, there are 10 possible patch states. Moreover, reduced

vectors and matrices (i.e.  $\widetilde{X}$ ) are obtained by removing all rows and  
488 columns that correspond to a patch state empty of the focal species and  
densities of free individuals from other species (so, for patch states, tilde  
490 vectors have  $\binom{N+S}{N} - \binom{N+S-1}{N} = \binom{N+S-1}{N-1}$  dimensions, following Pascal's  
rule);

(2) the equilibrium state  $P^*$  is not  $[1, 0, \dots, 0]$  in a metacommunity, but rather  
492 corresponds to the ecological equilibrium obtained when all other species  
494 are present and the focal species is absent;

(3) consequently, the initial state vectors  $Y_k$  are found by taking the columns  
496 of  $G$  that correspond to non-null elements of  $P^*$ , differentiating each of  
these columns with respect to  $M_k$  and averaging the resulting vectors  
498 following the weights found in  $P^*$ . Note that an initial state corresponds  
to a given  $M_k$  colonization rate, not to a particular patch state (i.e. it is  
500 not necessary to separate average initial states into a list of elementary  
initial states that are obtained with a certain probability, following  $P^*$ );

(4) the  $R_m$  criterion is then found as in the viability case, i.e. through deriving  
502 the  $R(0)$  matrix and looking for its dominant eigenvalue. Obviously, the  
504 larger size of the matrix will make it more difficult to get analytical  
expressions of  $R_m$ .

## 506 5.2 Does $R_m = 1$ hold when invading identical species?

For the models studied here,  $R_m$  is both an invasibility and a single-species  
508 persistence criterion. Persistence of the whole community is achieved when all  
species have  $R_m > 1$ . However, we have left unaddressed an important issue  
510 that only arises for invasibility and not for viability: is the  $R_m$  invasibility

criterion equal to one when a species tries to invade a community of like  
512 species? Like species, i.e. species that are identical in all relevant parameters,  
should behave as “neutral” species.

514 To simplify the problem, suppose that only two species are considered, a resi-  
dent species with phenotype  $X$ , and a potential invader with phenotype  $X'$ . In-  
516 vasion of the community by the potential invader happens when  $R_m(X', X) >$   
1 (the first variable indicates the invader’s phenotype, the second, the resi-  
518 dent’s). The problem of neutrality is: does  $R_m(X, X) = 1$  for all phenotypes  
 $X$ ? The property  $R_m(X, X) = 1$  is highly desirable as we expect  $R_m$  to behave  
520 as other fitness proxies (e.g.  $R_0$ ,  $r$ , etc.).

Metz and Gyllenberg (2001) hypothesized that this property holds in ecolog-  
522 ically sensible models, but no real proof of it exists. Moreover, we have found  
what looks like a counter-example: the metapopulation model developed by  
524 Slatkin (1974) to explain the regional coexistence of two species. In this model,  
exactly similar species (i.e. species for which all parameters in the equations  
526 are given the same value) can generally invade one another ( $R_m(X, X) > 1$ ).  
This odd result means mutual invasion of neutral species. Such “neutral in-  
528 vasions” do not occur in related models developed to address the effect of  
competition-colonization trade-off on species coexistence at the metacommuni-  
530 nity level (Hastings, 1980; Calcagno et al., 2006).

The key to this paradoxical situation is that Slatkin’s model is built under  
532 the assumption that the carrying capacity of a patch varies with the composi-  
tion of the community inhabiting the patch. Specifically, the model of Slatkin  
534 describes a situation in which the carrying capacity of patch is positively cor-  
related to the *number* of species in the patch. As the number of species has

536 nothing to do with the traits of this species, two “neutral” species still benefit  
from increased carrying capacity. Actually, there can be “mixed” patches in  
538 which all species have identical parameters.

The opposite situation can be conceived, for instance for organisms with a  
540 strong altruistic behavior directed only towards conspecifics. In such a model,  
the community carrying capacity would be negatively correlated to community  
542 diversity, yielding  $R_m(X, X) < 1$ .

These models both make the assumption that there is some way to discrim-  
544 inate species, even when these species are identical in all aspects (from the  
model’s point of view). Hence, there can be an advantage (as in Slatkin’s  
546 model) or a disadvantage (as in our hypothetical model) of being a rare  
species, irrespective of the traits considered in the model. This violates the  
548 symmetry in individual interactions required for neutrality, since individu-  
als discriminate heterospecifics from conspecifics (Chesson and Rees, 2007).  
550 Hence,  $R_m(X, X) \neq 1$ , but this is not a paradox. Still, it makes  $R_m$  unsuit-  
able as a fitness measure. It is likely that building a “sensible” model in which  
552  $R_m(X, X) = 1$  to address an evolutionary question is always possible (e.g. by  
adding a dummy trait addressing “species membership”, i.e. a “green beard”),  
554 but this remains to be clarified.

## 6 Discussion

### 556 6.1 *Essential results*

We have proved that the  $R_m$  criterion in continuous time (Gyllenberg and  
558 Metz, 2001; Metz and Gyllenberg, 2001) is equivalent to the general Jacobian  
criterion for instability of the trivial equilibrium. We have proved that the  
560 simple  $R_m$  formulation of Metz and Gyllenberg is valid only when coloniza-  
tion rates can be summarized using only one function of metapopulation state  
562 variables (e.g. disperser density, or the average of local population sizes, cf.  
the first two examples). When colonization rates depend on several functions,  
564 we derived a more general mathematical formulation of the  $R_m$  criterion. Al-  
though the original  $R_m$  criterion had not been presented for models with more  
566 than one class of initial states, we have shown it still applies (see also Parvinen  
and Metz, 2008 for an application to diploid genetics).

568 Our proof of the validity of the  $R_m$  criterion for one-species metapopulations  
(viability) can be generalized directly to invasibility in  $n$ -species metacommuni-  
570 nities. In metacommunities, the species  $R_m$  procedure consists in computing  
the mean number of new settlements due to each deme occupied by the focal  
572 species when it is rare, at a resident metacommunity steady state computed  
for the  $n - 1$  other species.

### 574 6.2 *Assumptions required for validity*

The  $R_m$  criterion has been shown to hold for at least two types of models:  
576 (i) discrete-time Markov models with Poisson-distributed immigration, and

binomial emigration (Chesson, 1984), and (ii) continuous-time Markov models  
578 (present study). The fact that this criterion holds in different contexts calls  
for a careful examination of the assumptions underlying its validity. We first  
580 delineate the key assumptions we made to prove the criterion, and then we  
compare the assumptions of this general class of models with those of models  
582 studied by Chesson (1984).

To prove the  $R_m$  criterion for continuous-time Markov metapopulation models,  
584 we have made six major assumptions:

- (1) colonization rates do not depend on the proportion of empty sites, and de-  
586 pend positively on all other metapopulation variables when the metapop-  
ulation is almost empty;
- 588 (2) metapopulation dynamics can be described by a matrix-based master  
equation (such as equation 10), i.e. there is an upper limit to the number  
590 of individuals present in a site.
- (3) the metapopulation consists of a large (nearly infinite) number of patches  
592 (in other words, we neglected demographic stochasticity at the metapop-  
ulation level);
- 594 (4) spatial structure was implicit (or equivalently, dispersal was uniform  
among demes);
- 596 (5) metapopulation dynamics are continuous, rather than discrete (i.e. there  
is no fixed time lag between events, events can happen any time);
- 598 (6) dynamics of the metapopulation are memoryless, i.e. the state of the  
metapopulation at time  $t$  only depends on its state during the immedi-  
600 ately previous time interval of length  $dt$ .

Two questions are essential here: (i) are these assumptions really necessary for

the  $R_m$  criterion to hold? (ii) are these assumptions “natural” and/or “realistic”?

*6.2.1 Assumption 1: colonization rates depend positively on all variables except empty patch frequency*

This assumption is (intuitively) necessary for the criterion to hold because we have explicitly used it (to prove that the  $r_{ij}$  are non-negative and decreasing in section A.7). In fact, it is possible that the  $R_m$  criterion still holds when this assumption is not true, but the corresponding proof is bound to be considerably more complicated than the present one. Anyway, this assumption appears quite realistic and natural. Its contrary would seem rather odd.

*6.2.2 Assumption 2: an upper limit to the number of individuals present in a site*

This may not be necessary since the Perron-Frobenius theorem, which is one of the main arguments behind the proof of the criterion, also holds for linear operator in infinite dimensions. However, some points of the proof may necessitate some changes to accomodate for this change in model assumptions. Besides, it seems quite natural to work with bounded local populations, just because the conservation of mass applies and restricts potential birth rate at the local scale when resources are limited (which is always the case).

*6.2.3 Assumption 3: a metapopulation consisting of a very large number of patches*

This assumption actually implies three things at the same time:

- 624 (1) averages equal means, i.e. theoretical expectations based on patch state  
probabilities are equal to empirical expectations based on actual patch  
626 states;
- (2) there is no demographic stochasticity at the metapopulation level (the  
628 state of the metapopulation never deviates from its mean-field expecta-  
tion, even though the state of each patch deviates stochastically from its  
630 mean-field expectation);
- (3) there is no risk of stochastic absorption in the empty metapopulation  
632 state (i.e. the metapopulation never gets empty when it started from a  
steady state non-empty situation if the species has  $R_m > 1$ ).

634 Relaxing this assumption would mean letting different effects affect the model.  
First, the absence of relation between average and mean implies that we cannot  
636 analytically resolve immigration pressures from the theoretical distribution of  
patch states. Instead, we have to focus on the empirical distribution of patch  
638 states, which may deviate significantly from its theoretical expectation when  
the number of patches is low. From a mathematical viewpoint, accepting this  
640 first effect involves at least switching from ordinary differential equations to  
stochastic differential equations. However, a more rigorous approach to this  
642 effect would necessitate accounting for the absence of continuity in the value  
of patch state probabilities (i.e. when the number of patch is  $K$ , the elemen-  
644 tary increment of patch state probabilities is  $1/K$ ). Second, the existence of  
important deviations of the metapopulation state around its expected mean  
implies a risk of accidental extinction (Nisbet and Gurney, 1982; Otto and  
646 Day, 2007). This risk is maximal when there is only one population and de-  
648 creases monotonically as the number of patches increases. The most simple  
and most realistic way to model a metapopulation that displays such an effect



650 is to account for the state of each and every population, and to count the  
number of individuals in each population (as in Lebreton, 1996). However,  
652 generalizations based on such models do not look easy.

Is this assumption realistic? In real systems, the actual number of populations  
654 that are part of the same connected system can be very important. However,  
when it is necessary to keep the assumption of uniform dispersal (see next  
656 section), the size of the metapopulation plays the role of the mean dispersal  
area and can be quite low. It is thus very helpful to simultaneously state  
658 assumptions on patch numbers and dispersal range distribution.

#### 6.2.4 *Assumption 4: dispersal is uniform*

660 Assuming that the metapopulation does not represent the range of populations  
over which dispersal can be approximated as uniformly distributed (see previ-  
662 ous section), but rather represents the total number of populations concerned  
by the model, it seems questionable to assume a uniform dispersal of offspring  
664 among patches. Indeed, this assumption is only realistic for organisms that  
undergo passive isotropic dispersal and have numerous fast-moving offspring  
666 to attain remote patches in a small increment of time. Moreover, another is-  
sue is linked to the existence of uniform dispersal, i.e. the fact that dispersal  
668 cost is uniform for all pairs of patches. Intuitively, long distance dispersal is  
expected to be more costly than dispersing to a neighboring patch.

670 A mathematical model that incorporates non-uniform dispersal is necessarily  
based on explicit spatial structure (see e.g. Neubert et al., 1995). When immi-  
672 gration in a certain patch only depends on the state of a few other patches, pair  
approximations can be used to lower the complexity of the problem and make

674 it analytically tractable. When populations are small and far apart, the model  
can be approximated by a point process in continuous space. In that context,  
676 a good approximation is to consider diffusion equations associated with the  
probability density  $p_k(x, t)$  of observing a population with  $k$  individuals at  
678 point  $x$  at time  $t$ . Such a probability density function obeys a Fokker-Planck  
equation that can be analytically studied and/or numerically solved for steady  
680 states. Although the  $R_m$  criterion may apply to such a model, its mathemat-  
ical formulation is not clear since the equivalent of the transition matrix is a  
682 linear operator on real-valued functions defined on space. Such a linear opera-  
tor has an infinite number of dimensions, so that its eigenvalues may be more  
684 difficult to obtain.

In the present paper, reproduction, mortality and migration have been much  
686 more detailed than the ecological interactions underlying these rates. Indeed,  
nothing is known about the competitive interactions that imply density-dependence  
688 and competition among individuals. Implicitly, competition is assumed to hap-  
pen only among individuals sharing the same patch. However, it may be that  
690 the scale at which individuals reproduce (i.e. the population scale) is different  
from the range at which competitive effects occur. In other words, offspring  
692 may be born in the same population as their parent(s), but individuals may  
compete for resources over several populations (thus implying that carrying  
694 capacities and density-dependence are shared over these populations). Taking  
into account these other scales (competition, predation, etc.) would complicate  
696 the model and make it deviate from its original “metapopulation” spirit. How-  
ever, it is worth mentioning that dispersal is only one of the processes whose  
698 spatial scale needs to be addressed in metapopulation (or metacommunity)  
models.

700 6.2.5 *Assumption 5: dynamics are continuous*

Mathematical models aimed at describing ecological situations are classically  
702 described as either continuous-time or discrete-time models. Here, we focused  
on certain continuous-time models to provide a proof of the  $R_m$  criterion.  
704 However, such a proof exists for some discrete-time models (Chesson, 1984).  
Thus, it does not seem that this assumption is necessary for the criterion to  
706 hold. As far as realism is concerned, discrete and continuous models can be  
argued for or against as easily. For instance, populations with pulsed birth  
708 events may be more realistically represented by discrete-time models, whereas  
populations of organisms with no stage structure and non-pulsed births are  
710 conveniently described by continuous-time models.

Relaxing the assumption of continuous dynamics affects the way immigration,  
712 emigration, birth and death are accounted for. In classical continuous-time  
models, these processes are described by their rates (this is the case in the  
714 present study), so that each event concerns one individual (one birth, one mi-  
grant or one death). In discrete-time models, these processes occur at fixed  
716 time steps. Thus modulating the strength of the different processes implies  
that an event must concern a variable number of individuals. For instance,  
718 in Chesson (1984), emigration of newborn offspring is binomial, i.e. each off-  
spring has a certain probability of emigrating at birth; in the same model,  
720 immigration follows a Poisson distribution, which is the natural limit for a  
binomial distribution ( $n$  trials,  $p$  probability of success) when the product  $n.p$   
722 (here, the ratio of immigrants divided by the number of patches) has a finit  
limit when  $n$  goes towards infinity.

724 It is worth remarking that events described in continuous time using a constant

rate or in discrete time using a Poisson distribution yield exactly the same dy-  
726 namics. Indeed, in the first case, the time between events is exponentially dis-  
tributed, so that averaging over a fixed time period, the number of events that  
728 occurred follows a Poisson distribution (the terms “Poisson process”, “Markov  
arrival process”, etc. actually stem from this equivalency). Hence, Chesson’s  
730 (1984) and our proofs actually show that any metapopulation model in which  
immigration is a Poisson process, be it in discrete or continuous time, obeys  
732 the  $R_m$  criterion.

### 6.2.6 Assumption 6: dynamics are memoryless

As a consequence of using a very simple master equation to describe the dy-  
namics of the metapopulation, the present class of models does not account for  
delayed effects or lags between migrant departure and arrival times. However,  
such a model can be envisaged as an extension of equation 10. If  $G_\tau$  describes  
the marginal effects (generalized “transition” matrix) imputable to the state  
of the metapopulation at a time  $t - \tau$  on the metapopulation at time  $t$ , the  
master equation could be written as:

$$\frac{dP}{dt}(t) = \int_0^\infty G_\tau[P(t - \tau)]P(t - \tau)d\tau \quad (63)$$

734 This equation is already a simplification of a more general problem since we  
have assumed that marginal effects only depend on time lag ( $\tau$ ) through the  
736 state of the metapopulation ( $P(t - \tau)$ ), and not on the actual time ( $t - \tau$ ).  
Equation 10 is recovered by letting  $G_\tau(P) = G(P)\delta_0(\tau)$  where  $\delta_0$  is Dirac’s  
738 delta. Intuitively, a metapopulation described by equation 63 responds differ-  
ently to the  $R_m$  criterion than more simple, memoryless models:

740 (1) it looks quite likely that the  $R_m$  computed using matrix  $G_0$  describes

well the initial growth of a sparsely occupied metapopulation: when the  
742 metapopulation receives its first immigrant, the long-term past is unim-  
portant;

744 (2) on the other hand, when this  $R_m$  value is superior to 1, the metapopu-  
lation may eventually ends in the extinct state due to delayed negative  
746 effects. For instance, if a metapopulation can overshoot its steady state  
mean density (due to, e.g., accelerating birth rates) and that overshooting  
748 this steady state mean density at time  $t$  causes deterministic extinction  
of the whole metapopulation at time  $t + T$ , then the metapopulation will  
750 periodically starts from nothing ( $R_m > 1$ ) and then fall to an empty  
state.

752 As the empty metapopulation state is an absorbing state when there is no  
extraneous immigration, the fitness of an initially rare species that obeys the  
754 above example should be 0. However, a very small extraneous immigration  
pressure creates a time-averaged metapopulation state quite different from  
756 the empty state. To resolve such a paradoxical situation, we need to take  
into account the existence of infinitesimal immigration from outside of the  
758 metapopulation, and to define weak persistence as the fact that an initially  
rare species becomes more common (on average over time). To assess this  
760 property we have two options:

(1) compute a long-term average fitness proxy for the metapopulation. For  
762 population models, this is usually done by computing the geometric aver-  
age of the  $R_0$  value over an infinite number of generations. However, such  
764 a long-term average requires that the fitness proxy can be used far from  
the trivial empty equilibrium. In the case of metapopulations, this trans-  
766 lates to the simple question: can metapopulation growth be represented

by the  $R_m$  when the metapopulation is already in a non-empty state? If  
768 this proves true, then the basic reproduction ratio of a metapopulation  
over a time period of length  $\tau$  would be obtained as  $\frac{1}{\tau} \exp \int_0^\tau \log R_m(t) dt$ .  
770 (2) compute the growth rate of the mean density of individuals in popula-  
tions,  $\langle N \rangle$ , using the fact that  $d\langle N \rangle / dt = [0, 1, 2, \dots] \cdot (dP/dt)$  (i.e. in-  
772 crease in the mean density can be computed using equation 63). This  
fitness proxy is usually less convenient than the former one because in  
774 most case this computation involves the knowledge of higher moments  
(variance, skewness, ...) of the individual density distribution.

776 In any case, the two roles that have been played by the  $R_m$  criterion in simple  
models, i.e. (i) assessing whether a lonely individual can invade a landscape  
778 and (ii) assessing whether a metapopulation is protected from determinis-  
tic extinction, are dissociated in memoryless models. Thus, a metapopulation  
780 might initially grow when rare but also be subject deterministic extinction  
once well developed, or, on the other hand, a metapopulation might be in-  
782 capable of starting from scratch but displays a steady state protected from  
deterministic extinctions (similarly to a population with an Allee effect).

### 784 6.3 Challenges and perspectives

The  $R_m$  criterion has not been proved in spatially explicit metapopulation  
786 model. A conceptual and mathematical puzzle now lies in the way spatially  
explicit metapopulations can be described in order to make them realistic  
788 while keeping the model analytically tractable.

Other complications, such as temporal variability, environmental heterogene-

ity, or stage-structure of individuals within populations, could also be incorporated in the model. Another formalism, developed by O. Diekmann and colleagues (Diekmann et al., 1998, 2001, 2003), and used in the more mathematically-oriented of Metz and Gyllenberg’s papers on the  $R_m$  criterion (Gyllenberg and Metz, 2001) could be used to construct the proof on fewer mathematical assumptions. It could also quite simply account for stage structure. Most interestingly, a proof built within this framework would be independent of the differential equation approach we used here.

In the two existing proofs of the  $R_m$  criterion’s validity, viability (growing when rare) and persistence (being protected from total extinction) are equivalent. However, several complications of the model can hinder this equivalence (e.g. incorporating demographic stochasticity, letting the dynamics depend on the state of the metapopulation in the distant past, etc.). Clearly, the definition of the  $R_m$  criterion is more suited for defining viability than persistence, i.e. a model can be conceived in which a rare species produces more emigrants than its immigrant input, and yet does not resist total extinction.

Once viability and persistence are clearly separated, the effect of demographic stochasticity on these two properties has to be assessed. It is quite possible that a metapopulation with  $R_m > 1$  is not viable due to the effect of demographic stochasticity. A well-known example of such an effect is given by studies on stochastic differential equations (Oksendal, 2007, chap. 5): when the growth rate of a population is the sum of a constant deterministic part  $r$  and of a random part  $\alpha W_t$  (where  $W_t$  is a white-noise process), the population grows exponentially only when  $r > \alpha^2/2$  (and not  $r > 0$  as predicted by purely deterministic models). When  $0 < r < \alpha^2/2$ , demographic stochasticity is strong enough to drive an exponentially growing population towards sure

816 extinction. An interesting improvement to the existing  $R_m$  criterion would be  
to account for this effect at the metapopulation scale.

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## **A Appendix: proof of the criterion**

### 826 *A.1 Notations*

The mathematical notations we use are presented in Table A.1.

828 With these notations, the invariant subspace of matrix  $M$  associated with  
eigenvalue  $\lambda$  is  $\ker(M - \lambda I)^{\nu(\lambda)}$ . By Jordan decomposition theorem (Horn and  
830 Johnson, 1985),  $\mathbb{C}^n = \bigoplus_{\lambda \in Sp(M)} \ker(M - \lambda I)^{\nu(\lambda)}$ .

A matrix is non-negative if all its elements are positive or zero.



Notation	Meaning
$X$	A vector
$x_i$	The $i$ th element of vector $X$
$[x_1, x_2, \dots, x_n]$	Vector $X$ written in row
$M$	A matrix
$m_{i,j}$	Element of matrix $M$ in row $i$ and column $j$
$m_{i,.} / m_{.,j}$	The $i$ th row / the $j$ th column vector of $M$
$X^T$	Transposed of $X$ , be it a vector or a matrix
$X.Y$	Scalar product of vectors $X$ and $Y$
$(E_i)$	Canonical orthonormal basis of $\mathbb{R}^n$
$\Omega$	First orthant of $\mathbb{R}^n$ ( $X$ vectors that verify $\forall i \in [1; n], x_i \geq 0$ )
$\text{Sp}(M)$	The set of eigenvalues of matrix $M$
$\text{Vect}(X)$	Vector subspace induced by vector $X$
$\ker(M)$	The kernel of matrix $M$ (subspace of vectors $X$ such that $M.X = 0$ )
$\rho(M)$	The spectral radius of matrix $M$ (highest modulus of its eigenvalues)
$\nu(\lambda)$	The index of eigenvalue $\lambda$ of a matrix
$\mathcal{M}_n(\mathbb{R})$	The space of square real matrices of dimension $n$
$f \circ g$	Composition of functions $f$ and $g$

Table A.1

Notations used in the article.  $n$  is a positive integer.

## 832 A.2 Context and definitions

We consider a closed spatially implicit metapopulation occupied by individuals from one species. We note  $P$  the vector that describes the state of the metapopulation, i.e. given first by all the probabilities that a sample patch is found containing 0, 1, 2,... individuals (possibly structured by classes, sex, etc.), and then by all densities of “free” individuals that do not reside in a patch (e.g. the density of dispersers if the model makes use of a disperser pool). The size of vector  $P$  is noted  $N + 1$ . We note  $G(P)$  the “transition matrix”, i.e. the matrix function that determines the master equation of  $P$ :

$$\frac{dP}{dt} = G(P).P \quad (\text{A.1})$$

Matrix  $G(P)$  contains the rates of birth, dispersal, death and extinction in the metapopulation. As any transition matrix,  $G$  has negative diagonal elements. The upper diagonal block of  $G$  determines the transition matrix of patch state probabilities. Thus, it is a conservative transition matrix, i.e. the sum of each of its column equals 0 (because the sum of probabilities remains equal to 1). The upper right-hand block of  $G$  is full of 0 (by construction: we model only flows among patch states mediated by densities of free individuals, not flows from densities of free individuals to patch states mediated by other patch states), its lower diagonal block describes the dynamics of the densities of free individuals (possibly in different classes), and its lower left-hand block describes the rates of production of free individuals in the different patch states. In general, the following relationships hold:

$$\forall i, g_{ii} < 0$$

$$\forall i, \forall j \neq i, g_{ij} \geq 0$$

$$\forall j, \sum_i g_{ij} \leq 0$$

where  $g_{ij}$  is the element of  $G$  on the  $i^{th}$  row and  $j^{th}$  column. As  $G$  depends on  
834  $P$ , each element  $g_{ij}$  is a function  $g_{ij}(P)$ . We assume that these functions are  
sufficiently smooth and derivable to allow the required differential expressions.  
836 In particular, the first column of  $G$ ,  $g_{.,1}$ , gives the transition rates from an  
empty patch state to other states. In the most simple models, only  $g_{2,1} \geq 0$   
838 (i.e. the only possible transition for an empty patch is to be colonized by one  
individual). More complicated models can be envisaged in which individuals  
840 can belong to different classes (e.g. wingless and winged individuals in aphids),  
so that more than one component of  $g_{.,1}$  may be positive. In any case, positive  
842 components of  $g_{.,1}$  correspond to colonization rates.

We assume that  $q$  of these colonization rates are positive, and these rates,  
844  $(M_k)_{k \in [1;q]}$ , are functions of  $P$ . We consider that the components of  $P$  are  
independent in the determination of  $M_k$  (i.e. the fact that the sum of state  
846 probabilities equates 1 is not used to simplify the expression for  $M_k$ ), so that  
 $\partial M_k / \partial p_0 = 0$  for all  $k$  (i.e. colonization rates do not depend directly on the  
848 proportions of empty patches). Moreover, we assume that all colonization rates  
respond positively to all increases of non-empty state probabilities or of den-  
850 sities of free individuals when the actual metapopulation is completely empty  
(state  $P^* = [1, 0, 0, \dots, 0]$ ), i.e. for all  $k$ , the vector  $\partial_P M_k$  taken at  $P = P^*$  has  
852 only non-negative components.

### A.3 Linearization of the master equation

854 Around an equilibrium  $P^*$  of equation A.1, the dynamics of small deviations  
of  $P$  can be linearized using the differential form  $\partial_P G$  of  $G$ :

$$\frac{dP}{dt} \approx G(P^*).P + \partial_P G(P^*)(P - P^*).P^* \quad (\text{A.2})$$

856  $\partial_P G(P^*)$  is a linear function  $\mathbb{R}^{N+1} \mapsto \mathcal{M}_{N+1}(\mathbb{R})$ . The viability of the metapopulation can be studied through the instability of Eq. A.2 near the special equilibrium in which all demes are empty ( $P^* = [1, 0, 0, \dots, 0]$ ). In this case, the  
858 only components of  $\partial_P G(P^*)(P - P^*)$  that will be used to compute  $\frac{dP}{dt}$  are  
860 the ones associated with the first column of  $G(P)$ , as it can be seen in the following development:

$$\begin{aligned} \partial_P G(P^*)(P - P^*).P^* &= \left( \sum_{i=0}^N (p_i - p_i^*) \partial_{p_i} G(P^*) \right).P^* \\ \Rightarrow \partial_P G(P^*)(P - P^*)P^* &= \sum_{i=0}^N (p_i - p_i^*) \partial_{p_i} g_{.,1}(P^*) \end{aligned} \quad (\text{A.3})$$

As we have assumed that  $g_{.,1}$  depends on only  $q$  variables (the  $M_k$ ), we can further develop  $\partial_P G(P^*)dP.P^*$ :

$$\partial_P G(P^*)dP.P^* = \sum_{k=1}^q \partial_{M_k} g_{.,1}(P^*) \circ \partial_P M_k.dP \quad (\text{A.4})$$

Because  $g_{.,1}$  is a vector function of  $q$  real variables, partial differentials,  $\partial_{M_k} g_{.,1}(P^*)$ , are vectors.  $\partial_P M_k$  is a vector, noted  $A_k$  (which has only non-negative components, cf. section A.2). Let  $Y_1, Y_2, \dots, Y_q$  be the vectors defined by:

$$\forall k, Y_k = \partial_{M_k} g_{.,1}(P^*) \quad (\text{A.5})$$

Eq. A.2 can be rewritten as:

$$\frac{dP}{dt} \approx G(P^*).P + \sum_{k=1}^q (A_k.P)Y_k \quad (\text{A.6})$$

Instability of Eq. A.6 is achieved if and only if the Jacobian matrix  $J(P^*)$  defined by

$$J(P^*) = G(P^*) + \sum_{k=1}^q Y_k A_k^T \quad (\text{A.7})$$

862 has at least one eigenvalue with positive real part.

As we consider a closed metapopulation, colonization rates equal 0 when  
864 all patches are empty. In mathematical terms, this statement amounts to  
 $g_{.,1}(P^*) = 0$ . Because the first column of  $Y_k A_k^T$  is a null vector (the first com-  
866 ponents of  $A_k$  is equal to  $\partial M_k / \partial p_0$ , which equals 0), it follows that matrix  
 $J(P^*)$  is block upper triangular. The first diagonal element of  $G(P^*)$ , which  
868 corresponds to the first eigenvalue of  $J(P^*)$ , is 0. Thus, proving the strict in-  
stability of Eq. A.6 is equivalent to studying whether the lower diagonal block  
870 of  $J(P^*)$  has an eigenvalue with positive real part.

To that end, we introduce the tilde operator, which is defined as follows: if  $M$  is a  $N + 1$  square matrix,  $\widetilde{M}$  is defined as the lower  $N \times N$  diagonal block of  $M$  (obtained by removing its first row and first column). Similarly, if  $V$  is a  $N + 1$  vector,  $\widetilde{V}$  is defined as the vector obtained from  $V$  by removing its first element. Matrix  $\widetilde{J}$  ( $= \widetilde{J}(P^*)$ ) verifies the following relation:

$$\widetilde{J} = \widetilde{G} + \sum_{k=1}^q \widetilde{Y}_k \widetilde{A}_k^T \quad (\text{A.8})$$

Using these notations, viability of the metapopulation is achieved if, and only if, matrix  $\widetilde{J}$  has at least one eigenvalue with positive real part. The reduced version of Eq. A.6 is:

$$\frac{d\widetilde{P}}{dt} \approx \widetilde{J}.\widetilde{P} = \widetilde{G}.\widetilde{P} + \sum_{k=1}^q (\widetilde{A}_k.\widetilde{P})\widetilde{Y}_k \quad (\text{A.9})$$

We shall now derive necessary and sufficient conditions for viability of the

872 metapopulation. The next sections present the mathematical argument leading  
to the " $R_m$  theorem". Some technical details are given in Appendix B. From  
874 now on, all dimensions of matrix and vector spaces are  $N$  (e.g.  $\Omega$  is the first  
orthant of  $\mathbb{R}^N$ ).

876 **Remark 2** *We note that the vectors  $\widetilde{Y}_k$  are independent. Indeed, each  $Y_k$  cor-  
responds to the derivative of  $g_{.,1}$  by one of the  $M_k$ , which in turn applies to a  
878 specific colonization transition.*

**Remark 3** *We note that  $\widetilde{Y}_k$  and  $\widetilde{A}_k$  are in  $\Omega$  (i.e. have only non-negative  
880 components). This is a result of the construction of  $Y_k$  (equation A.5) and a  
“natural” constraint on the relationship between state probabilities, free indi-  
882 vidual densities and migration pressure at the equilibrium (section A.2).*

**Remark 4** *If  $X \in \Omega$  and  $Y \in \Omega$ ,  $X.Y \geq 0$ . This is straightforward: all vectors  
884 in  $\Omega$  have only non-negative components, so that scalar products between two  
elements of  $\Omega$  are non-negative.*

#### 886 A.4 Uniqueness of the dominant eigenvalue

Differential equation A.9 has all its solutions of the form  $t \mapsto e^{t\tilde{J}}.V$ . In ap-  
888 pendix B, we prove that all these solutions stay in  $\Omega$  if they begin in it. The  
following proposition, based on results given in Appendix B reduces the prob-  
890 lem of looking for eigenvalues of  $\tilde{J}$  with positive real parts.

**Proposition 5**  *$\tilde{J}$  only admits one eigenvalue with highest real part, and this  
892 eigenvalue is real.*

**PROOF.** From proposition 17 given in Appendix B applied to  $t = 1$ , there is  
894 an eigenvalue of  $e^{\tilde{J}}$  equal to  $\rho(e^{\tilde{J}})$ . From lemma 19 given in Appendix B, there  
is an eigenvalue  $\lambda$  of  $\tilde{J}$  such that  $\rho(e^{t\tilde{J}}) = e^{\lambda}$ . For all other eigenvalues  $\mu$  of  $\tilde{J}$ ,  
896  $e^{\mu}$  is an eigenvalue of  $e^{\tilde{J}}$ , and thus, from proposition 17,  $e^{\Re(\mu)} < e^{\Re(\lambda)}$  which  
proves that  $\lambda$  has the highest real part of all elements of  $Sp(\tilde{J})$ . The same line  
898 of reasoning leads to conclude that for all non-negative  $t$ ,  $\rho(e^{t\tilde{J}}) = e^{\lambda t}$ . From  
proposition 17, this means that  $e^{\lambda t}$  is real for all  $t \geq 0$ , or equivalently that  
900  $\Im(\lambda t) \equiv 0 [2\pi]$  for all  $t \geq 0$ . This last statement implies that  $\Im(\lambda) = 0$ , and  $\lambda$   
must be real. Uniqueness of  $\lambda$  is also achieved since all candidates  $\lambda$  must be  
902 real, so that if their real parts are equal, eigenvalues are equal.  $\square$

#### A.5 Invertibility of the reduced transition matrix

904 **Proposition 6** *If  $\lambda \geq 0$ ,  $\tilde{G} - \lambda I$  is invertible.*

**PROOF.** We remark that  $\tilde{G}$  is invertible because:

906 (i) its upper diagonal block (patch state probabilities) obeys Hadamard's the-  
orem (its diagonal elements are negative and greater, in absolute value, than  
908 the sum of all non-diagonal elements in each column),

(ii) its lower diagonal block (densities of free individuals) has only strictly  
910 negative components on its diagonal (the effect of mortality and outward flow  
rates).

912 Using the same argument, we conclude that  $\tilde{G} - \lambda I$  is invertible for all positive  
 $\lambda$ .  $\square$

914 **Corollary 7**  $\tilde{G}$  has no positive eigenvalue.

### A.6 Transformation of the eigenvalue problem

Consider the eigenvalue problem defined by  $\tilde{J}.V = \lambda V$ . This problem is equivalent to finding  $\lambda$  and  $V$  such that  $(\tilde{J} - \lambda I).V = 0$ , i.e. whether  $0 \in \text{Sp}(\tilde{J} - \lambda I)$ . In the previous section, we have shown that  $\tilde{G} - \lambda I$  is invertible when  $\lambda \geq 0$ . For non-negative  $\lambda$ , let  $Z_k(\lambda)$  be defined as:

$$Z_k(\lambda) = -(\tilde{G} - \lambda I)^{-1} \tilde{Y}_k \quad (\text{A.10})$$

916 **Remark 8** As a corollary of proposition 20 and remark 3, we note that all scalar products  $\tilde{A}_i.Z_j(\lambda)$  are non-negative (cf. remark 4)..

The following identity:

$$\tilde{J} - \lambda I = (\tilde{G} - \lambda I) \cdot \left( I - \sum_{k=1}^q Z_k(\lambda) \tilde{A}_k^T \right) \quad (\text{A.11})$$

insures that, for all non-negative  $\lambda$ :

$$\lambda \in \text{Sp}(\tilde{J}) \Leftrightarrow 1 \in \text{Sp} \left( \sum_{k=1}^q Z_k(\lambda) \tilde{A}_k^T \right) \quad (\text{A.12})$$

We now define matrix  $H(\lambda)$  (or more simply  $H$ ):

$$H(\lambda) = \sum_{k=1}^q Z_k(\lambda) \tilde{A}_k^T \quad (\text{A.13})$$

Equation A.12 can be reformulated as:

$$\lambda \in \text{Sp}(\tilde{J}) \Leftrightarrow 1 \in \text{Sp} [H(\lambda)] \quad (\text{A.14})$$



918 *A.7 Reduction of the  $H(\lambda)$  matrix*

Let  $\Xi$  be the vector subspace induced by the family of  $Z_k(\lambda)$  vectors, i.e.  $\Xi = \text{Vect}(Z_1, \dots, Z_q)$ . Following remark 2, the family of  $Z_k(\lambda)$  vectors is independent, and thus  $\dim \Xi = q$ . We remark that  $\text{Im}(H) \subset \Xi$ , so that  $H.\Xi \subset \Xi$ . Thus, we may define the reduction  $H_\Xi$  of  $H$  on  $\Xi$ . This also implies that all eigenvectors of  $H$  associated with non-null eigenvalues are elements of  $\Xi$ . We note  $R(\lambda)$  (or more simply  $R$ ) the  $q \times q$  matrix representing  $H_\Xi$  on the basis formed by vectors  $Z_1, Z_2, \dots, Z_q$ . Elements  $r_{ij}(\lambda)$  of  $R(\lambda)$  are given by the scalar product:

$$r_{ij}(\lambda) = \widetilde{A}_i \cdot Z_j(\lambda) \quad (\text{A.15})$$

**Proposition 9** *For all  $i$  and  $j$ , and as long as  $\lambda$  remains non-negative,  $r_{ij}$  is*  
920 *a non-negative decreasing function of  $\lambda$ . Besides,  $r_{ij}(\lambda) \rightarrow 0$  when  $\lambda \rightarrow \infty$ .*

**PROOF.** All  $r_{ij}$  are non-negative (cf. 8). Now

$$r_{ij}(\lambda) = -\widetilde{A}_i \cdot (\widetilde{G} - \lambda I)^{-1} \widetilde{Y}_j \quad (\text{A.16})$$

so that

$$\frac{dr_{ij}}{d\lambda} = -\widetilde{A}_i \cdot (\widetilde{G} - \lambda I)^{-2} \widetilde{Y}_j \quad (\text{A.17})$$

Because  $\widetilde{A}_i \in \Omega$ ,  $-\widetilde{A}_i (\widetilde{G} - \lambda I)^{-1}$  and  $^{-2}$  are also in  $\Omega$  (cf. proposition 20  
922 in Appendix B). Given that  $\widetilde{Y}_j \in \Omega$ ,  $\widetilde{A}_i \cdot (\widetilde{G} - \lambda I)^{-2} \widetilde{Y}_j \geq 0$  (cf. remark 4),  
so that  $\frac{dr_{ij}}{d\lambda} \leq 0$ . Convergence of  $r_{ij}$  towards 0 is insured by proposition 21  
924 (Appendix B).  $\square$

**Proposition 10** *The dominant eigenvalue of matrix  $R$ ,  $m(\lambda) = \rho[R(\lambda)]$ , is*  
926 *real, non-negative and converges to 0 as  $\lambda \rightarrow \infty$ . It is a continuous function*

of  $\lambda$ .

**PROOF.** From proposition 9,  $R(\lambda)$  is non-negative, so that the Perron-Frobenius theorem applies, i.e.  $m(\lambda)$  is real and

$$0 \leq \min_i \sum_j r_{ij}(\lambda) \leq m(\lambda) \leq \max_i \sum_j r_{ij}(\lambda) \quad (\text{A.18})$$

928 which proves almost all results (cf. proposition 9).  $m$  is clearly continuous since all  $r_{ij}$  are continuous functions of  $\lambda$ .  $\square$

930 In order to prove that  $m$  is a decreasing function of  $\lambda$ , we need the following lemma:

932 **Lemma 11** *Let  $\Phi$  be the set of decreasing functions of  $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ . Let  $f$  and  $g$  be elements of  $\Phi$ . The following propositions hold:*

934 (i)  $f + g \in \Phi$ ;

(ii)  $fg \in \Phi$ ;

936 (iii)  $\forall \alpha > 0, f^\alpha \in \Phi$ .

**PROOF.** The proof of it is straightforward.  $\square$

938 **Corollary 12** *For all  $k \geq 1$ , all elements of  $R(\lambda)^k$  are non-negative decreasing functions of  $\lambda$ .*

940 We can now prove that:

**Proposition 13**  *$m$  is a decreasing function of  $\lambda$ .*

**PROOF.** We apply Gelfand's formula to  $m$ :

$$m(\lambda) = \lim_{k \rightarrow \infty} \|R(\lambda)^k\|^{1/k} \quad (\text{A.19})$$

942 for all matrix norm  $\|\cdot\|$ , i.e. the series of functions  $m_k = \|R^k\|^{1/k}$  converges  
(pointwise) to  $m$ . Using lemma 11 and corollary 12, all functions  $m_k$  are non-  
944 negative and decreasing. Finally, pointwise convergence insures that  $m$  is de-  
creasing.  $\square$

#### 946 A.8 The final step

Let  $R_m$  be defined as:

$$R_m = m(0) \quad (\text{A.20})$$

The “ $R_m$ ” theorem can be stated as:

948 **Theorem 14**  $R_m > 1 \Leftrightarrow \text{Sp}(\tilde{J}) \cap \mathbb{R}_+^* \neq \emptyset$ .

**PROOF.** If  $R_m > 1$ , there exists  $\lambda > 0$  such that  $m(\lambda) = 1$  (because  $m$  is de-  
950 creasing, continuous and converges towards 0 as its variable tends to infinity).  
Thus,  $1 \in \text{Sp}[H(\lambda)]$  and, following equation A.14,  $\lambda \in \text{Sp}(\tilde{J})$ . Reciprocally, if  
952  $\lambda \in \text{Sp}(\tilde{J}) \cap \mathbb{R}_+^*$ , then (equation A.14)  $1 \in \text{Sp}[H(\lambda)]$ . Thus,  $m(\lambda) \geq 1$  and, as  $m$   
is decreasing,  $R_m \geq 1$ . However, if  $R_m = 1$ , then all  $\mu \in [0; \lambda]$  are eigenvalues  
954 of  $\text{Sp}(\tilde{J})$ , which is finite. Hence,  $R_m > 1$ .  $\square$

## B Appendix: useful intermediate results

### 956 B.1 Boundedness of Eq. A.9 solutions

Differential equation A.9 has all its solutions of the form  $t \mapsto e^{t\tilde{J}}.V$ . To show  
958 that all these solutions stay in  $\Omega$  if they begin in it, we show the following  
proposition:

960 **Proposition 15** *For all  $i \in [1; N]$  and for all  $V \in \Omega$ , if  $E_i.V = 0$ , then  
 $E_i.\tilde{J}V \geq 0$ .*

**PROOF.** Suppose that  $V = (v_i)$  is a vector on the boundary of  $\Omega$  such that  
 $v_i = E_i.V = 0$ .  $\tilde{J}V$  can be developed:

$$\tilde{J}V = \tilde{G}V + \sum_{k=1}^q \tilde{A}_k.V\tilde{Y}_k \quad (\text{B.1})$$

so that

$$E_i.\tilde{J}V = E_i.\tilde{G}V + \sum_{k=1}^q (\tilde{A}_k.V)(E_i.\tilde{Y}_k) \quad (\text{B.2})$$

The second series of terms in the above sum is positive since the  $\tilde{A}_k$  and  $\tilde{Y}_k$   
are in  $\Omega$  (remark 4). The first term develops as follows:

$$E_i.\tilde{G}V = \sum_k \tilde{g}_{ik}v_k \quad (\text{B.3})$$

962 Because  $G$  is pseudo-stochastic, we have  $\tilde{g}_{ii} \leq 0$ , and  $\forall j \neq i, \tilde{g}_{ij} \geq 0$ . As  $v_i = 0$   
and  $\forall j, v_j \geq 0$ , it follows that  $E_i.\tilde{G}V \geq 0$ .  $\square$

### 964 B.2 Positivity property

**Proposition 16** *For all non-negative  $t$ , matrix  $e^{t\tilde{J}}$  is non-negative.*

966 **PROOF.** Proposition 15 means that for all  $V$  on the boundary of  $\Omega$ , the  
speed vector of differential equation A.9 points inwards. This entails that for  
968 all  $V$  in  $\Omega$  and for all non-negative  $t$ ,  $e^{t\tilde{J}}V$  stays in  $\Omega$ . Taking the canonical  
basis  $(E_i)$ , which is in  $\Omega$ , it follows that all columns of  $e^{t\tilde{J}}$  are in  $\Omega$ , and thus  
970  $e^{t\tilde{J}}$  is non-negative.  $\square$

**Proposition 17** *For all non-negative  $t$ ,  $\rho(e^{t\tilde{J}})$  is a non-negative real eigen-*  
972 *value of  $e^{t\tilde{J}}$ . In particular, for all  $\lambda$  in  $\text{Sp}(e^{t\tilde{J}}) \setminus \{\rho(e^{t\tilde{J}})\}$ ,  $|\lambda| < \rho(e^{t\tilde{J}})$ .*

**PROOF.** The proof results from the application of the Perron-Frobenius the-  
974 orem to  $e^{t\tilde{J}}$ , which is non-negative.  $\square$

### B.3 A spectral mapping lemma

976 **Proposition 18** *If  $\lambda \in \mathbb{C}$  is an eigenvalue of matrix  $M$  and  $V \in \mathbb{C}^n$  an*  
*associated eigenvector,  $V$  is also an eigenvector of matrix  $e^M$  associated with*  
978 *eigenvalue  $e^\lambda$ .*

**PROOF.** If  $MV = \lambda V$ , then

$$e^M V = \sum_{k=0}^{+\infty} \frac{M^k V}{k!} = \sum_{k=0}^{+\infty} \frac{\lambda^k V}{k!} = e^\lambda V \quad (\text{B.4})$$

which proves the result.  $\square$

**Lemma 19** *For all  $t$ , invariant subspaces of  $e^{tM}$  in  $\mathbb{C}^n$  are subspaces of in-*  
*variant subspaces of  $M$ . Besides, for all  $t$ , we have*

$$\text{Sp}(e^{tM}) = \{e^{\lambda t} | \lambda \in \text{Sp}(M)\} \quad (\text{B.5})$$

980 **PROOF.** The exponential of a block diagonal matrix is still a block diagonal  
matrix. If we decompose  $M$  using Jordan normal form, its decomposition is  
982 block diagonal and defines its invariant subspaces (Horn and Johnson, 1991).  
Thus, invariant subspaces of  $e^{tM}$  are subspaces of the invariant subspaces of  
984  $M$ . From proposition 18, the  $\supset$  inclusion of relation B.5 is known to be true.  
The reverse inclusion comes from the fact that the exponential of a Jordan  
986 block has diagonal terms equal to  $e^{\lambda_i t}$  where  $\lambda_i$ 's are eigenvalues of  $M$  (see  
e.g. Horn and Johnson, 1991), and thus, using the characteristic polynomial  
988 of  $e^{tM}$ , we can prove the lemma.  $\square$

#### B.4 Stability of $\Omega$ through $(\tilde{G} - \lambda I)^{-1}$

**Proposition 20** *If  $\lambda \geq 0$  and  $W \in \Omega$ , (i.e.  $\forall i, z_i \geq 0$ ), the solution vector  $W'$  to the equation*

$$W'^T = -W^T (\tilde{G} - \lambda I)^{-1} \quad (\text{B.6})$$

990 *is in  $\Omega$ .*

**PROOF.** Let  $w'_i$  be the elements of vector  $W'$  and  $g_{ij}$  the elements of  $\tilde{G}$ .  
 $W'^T (\tilde{G} - \lambda I) = -W^T$  can be decomposed for each column  $j$ :

$$(g_{jj} - \lambda)w'_j + \sum_{i \neq j} g_{ij}w'_i = -w_j \quad (\text{B.7})$$

Given that all diagonal elements  $g_{jj}$  of  $\tilde{G}$  are strictly negative (due to extinction and mortality rates),  $\lambda - g_{jj} > 0$  and thus Eq. B.7 can be further modified:

$$w'_j - \sum_{i \neq j} \left( \frac{g_{ij}}{\lambda - g_{jj}} \right) w'_i = \frac{w_j}{\lambda - g_{jj}} \quad (\text{B.8})$$

Now let  $\chi_j = \sum_{i \neq j} g_{ij} w'_i$ . Multiplying Eq. B.8 by  $g_{jk}$  and summing over  $j$ , we get for all  $k$ :

$$\chi_k - \sum_{j \neq k} \left( \frac{g_{jk}}{\lambda - g_{jj}} \right) \chi_j = \sum_{j \neq k} \frac{g_{jk} w_j}{\lambda - g_{jj}} \geq 0 \quad (\text{B.9})$$

Let  $\chi_j = \min_k(\chi_k)$ . Then, Eq. B.9 implies that:

$$\chi_j \left( 1 - \sum_{j \neq k} \left( \frac{g_{jk}}{\lambda - g_{jj}} \right) \right) \geq 0 \quad (\text{B.10})$$

However, the sum of each column of  $\tilde{G}$  is negative, and thus  $\chi_j \geq 0$ . If we  
992 inject this result into Eq. B.8, the proposition is then finally proved.  $\square$

### B.5 Convergence of $(\tilde{G} - \lambda I)^{-1}$

994 **Proposition 21**  $\left\| (\tilde{G} - \lambda I)^{-1} \right\| \rightarrow 0$  when  $\lambda \in \mathbb{R}^+ + i\mathbb{R}$  and  $\Re(\lambda) \rightarrow \infty$ .

**PROOF.** First, assume that  $\tilde{G}$  is diagonalizable in  $\mathbb{C}$ . Let  $Q$  be the invertible complex matrix which changes  $\tilde{G}$  into its Jordan normal form, i.e.

$$Q^{-1} \tilde{G} Q = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \quad (\text{B.11})$$

where  $\lambda_i$ 's all have negative real parts.

Let  $\lambda$  be a complex number with positive real part. From Eq. B.11, we obtain:

$$\tilde{G} - \lambda I = Q \begin{bmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \lambda_n - \lambda \end{bmatrix} Q^{-1} \quad (\text{B.12})$$

and thus

$$(\tilde{G} - \lambda I)^{-1} = Q^{-1} \begin{bmatrix} \frac{1}{\lambda_1 - \lambda} & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \frac{1}{\lambda_n - \lambda} \end{bmatrix} Q \quad (\text{B.13})$$

From Eq. B.13 and using the definition of the spectral norm (which is consistent with the natural scalar product on vectors), we immediately deduce that

$$\|(\tilde{G} - \lambda I)^{-1}\| \leq \|Q^{-1}\| \sqrt{\max \left| \frac{1}{\lambda_i - \lambda} \right|^2} \|Q\| \quad (\text{B.14})$$

Now we observe that

$$\frac{1}{\lambda_i - \lambda} = \frac{\Re(\lambda_i) - \Re(\lambda) - i(\Im(\lambda_i) - \Im(\lambda))}{(\Re(\lambda_i) - \Re(\lambda))^2 + (\Im(\lambda_i) - \Im(\lambda))^2} \sim \frac{-1}{\Re(\lambda)} \quad (\text{B.15})$$

and thus

$$\|(\tilde{G} - \lambda I)^{-1}\| \rightarrow 0 \quad (\text{B.16})$$

996 when  $\Re(\lambda) \rightarrow \infty$ . As this result is true for all diagonalizable matrices, it also  
holds for all possible  $\tilde{G}$  matrices given that  $M \mapsto \|(M - \lambda I)^{-1}\|$  is continuous  
998 on  $\tilde{G}$  matrices and owing to the density of diagonalizable matrices in complex  
matrices.  $\square$



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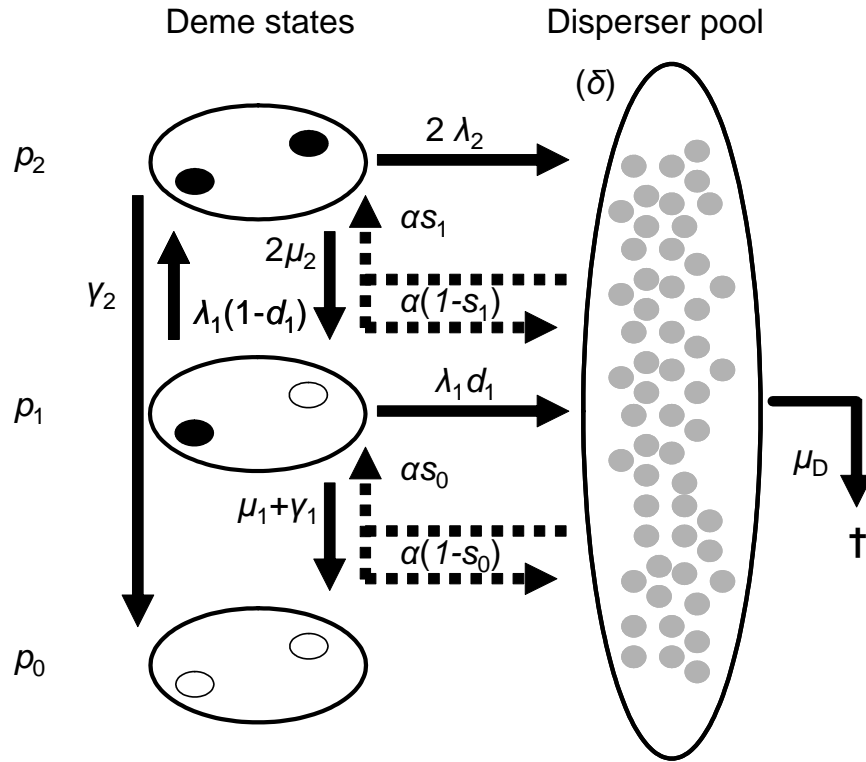


Figure 1. Schematic representation of Metz and Gyllenberg's model when  $N = 2$ . Solid arrows represent flows proportional to the source, dotted arrows represent flows proportional to the source times the density of dispersers. Symbols are as in the text.

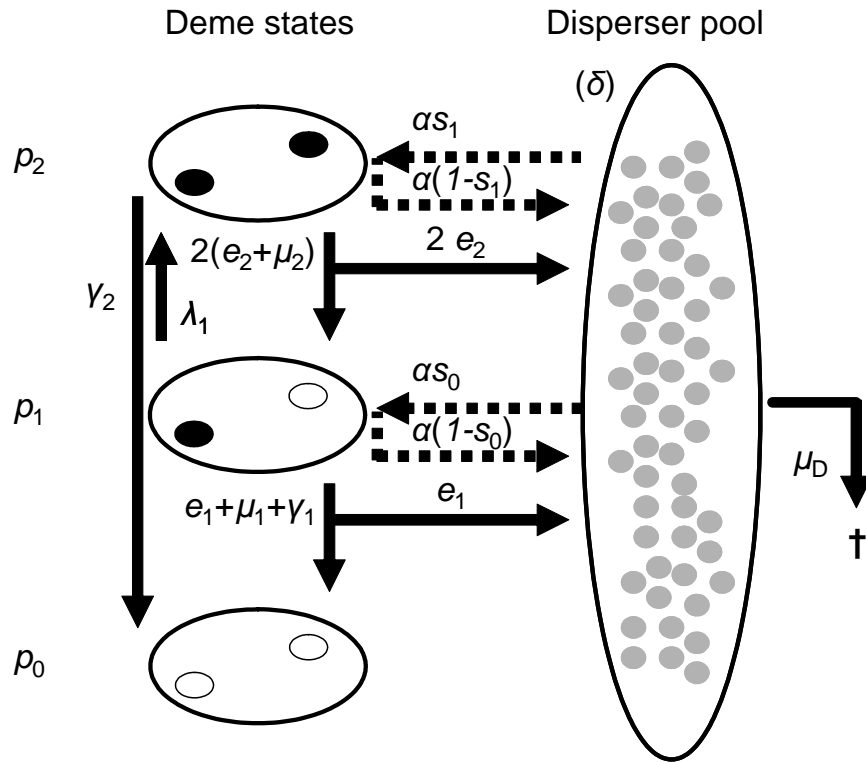


Figure 2. Schematic representation of the animal model with mobile adults. Solid arrows symbolize flows proportional to the source, dotted arrows represent flows proportional to the source times the density of dispersers. Other indications are as in the text.

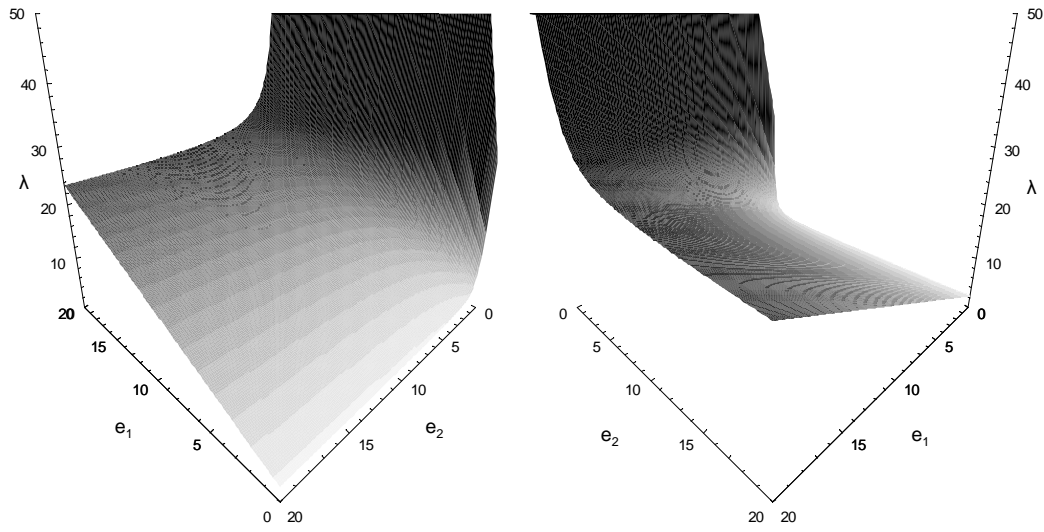


Figure 3. Surface defined by  $R_m = 1$  in the three-parameter space  $(e_1, e_2, \lambda)$  of the mobile adult model. The regions of viability is above the surface. The two panels represent the same surface observed from two different viewpoints. Grey shades indicate isopleths of  $\lambda$  (darker shades indicate higher  $\lambda$ ). Parameter values are:  $\alpha = 1$ ,  $\mu = 1$ ,  $\gamma = 0.1$ ,  $\mu_D = 1$ ,  $s_0 = 1$ .

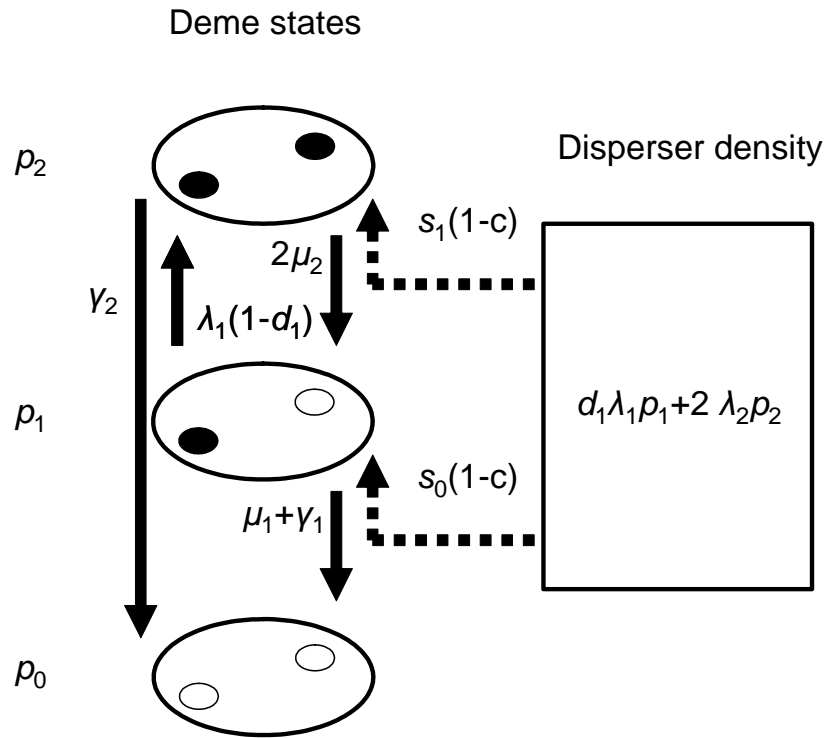


Figure 4. Schematic representation of the plant model with instantaneous dispersal. Solid arrows symbolize flows proportional to the source, dotted arrows represent flows proportional to the source times the steady-state density of dispersers. Other indications are as in the text.



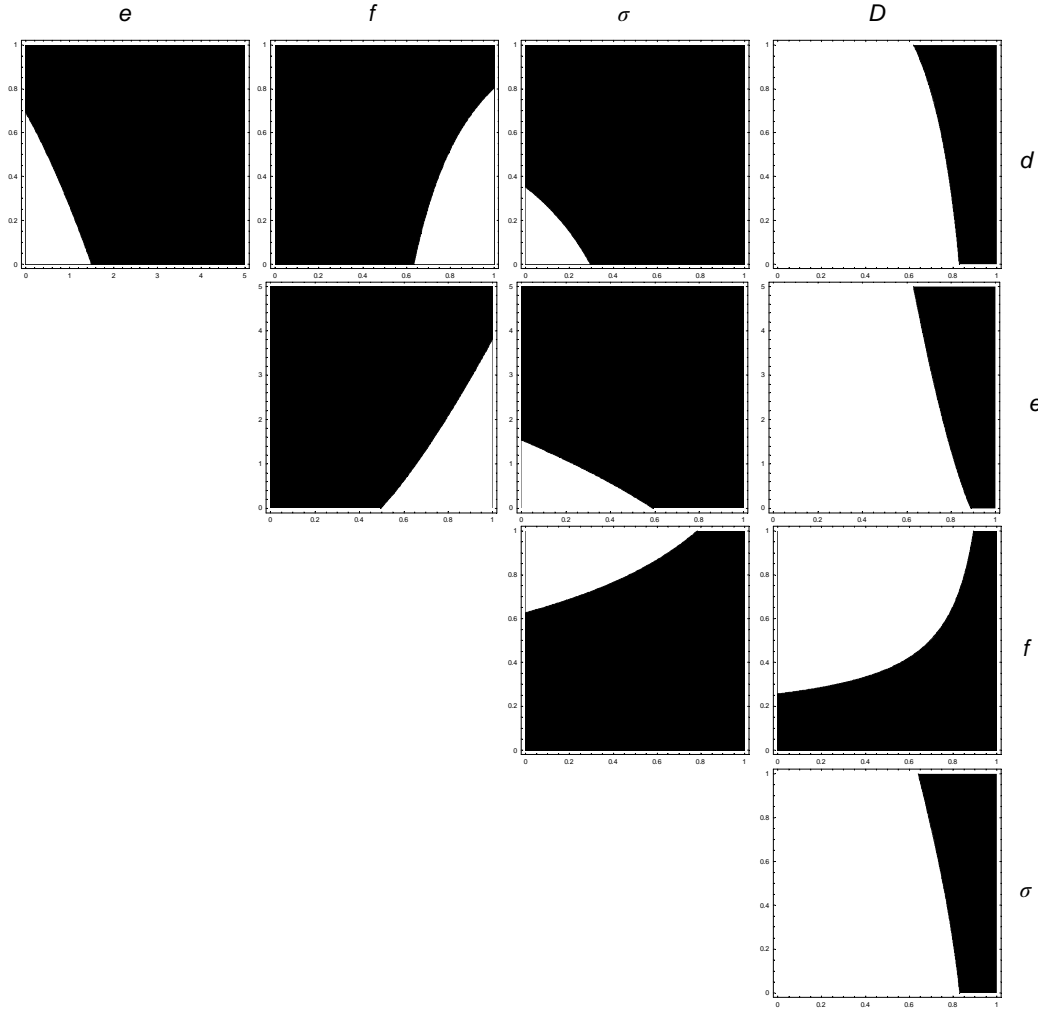


Figure 5. Two-varying parameter plots of the zones where  $R_m > 1$  (white regions) for the model with hermaphrodites and males. Default parameter values:  $d = 0.1$ ,  $e = 1$ ,  $f = 0.7$ ,  $\sigma = 0.1$ ,  $D = 0.8$ . Other parameters:  $\gamma = 0.1$ ,  $s_0 = 1$ ,  $c = 0.1$ ,  $\mu_H = \mu_M = 1$ ,  $\lambda = 5$ .